Collocation methods for second-kind Volterra integral equations with weakly singular kernels

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In this paper it is shown that the use of uniform meshes leads to optimal convergence rates provided that the analytic solutions of a particular class of Volterra integral equations (VIEs) are smooth. If the exact solutions are not smooth, however, suitable transformations can be made so that the new VIEs possess smooth solutions. Spline collocation methods with uniform meshes applied to these new VIEs are then shown to be able to yield optimal (global) convergence rates. The general theory is applied to a typical case, i.e. the integral kernels consisting of the singular term $(t-s)^{-\frac{1}{2}}$.

1. Introduction

This paper will be concerned with high-order collocation methods for the second-kind Volterra integral equations (VIEs)

$$y(t) = \int_0^t p(t,s)k(t,s,y(s))\,ds + g(t), \qquad t \in I(T) := [0,T] \tag{1.1}$$

where $y(t)$ is the unknown function whose value is to be determined in the interval $0 \leqq t \leqq T < \infty$, the kernel $k(t,s,y(s))$ is Lipschitz continuous in its third variable and $p(t,s)$ is unbounded in the region of integration but integrable over $I(T)$. Such equations have received a considerable amount of attention both numerically and theoretically (see, e.g. Brunner and van der Houwen [**5**], who provide a comprehensive list of references).

The following notation and methods were introduced in [**3**,**4**] and will be used throughout this paper. The collocation methods generate, as approximations to the solution of (1.1), elements of the polynomial spline space

$$S^{(d)}_{m-1}(Z_N,T) := \{u \in C^{(d)}(I(T)) : u\,|_{\sigma_n} := u_n \in \pi_{m-1}, 0 \leqq n \leqq N-1\}, \tag{1.2}$$

associated with a given partition

$$\Pi_N \;:\; 0 = t_0 < t_1 < \cdots < t_N = T, \qquad N \geqq 1 \tag{1.3}$$

of the interval $I(T)$. Here, π_{m-1} is the set of (real) polynomials of degree not exceeding $m-1$ (with $m \geqq 1$), and we have set $\sigma_0 := [t_0, t_1]$ and $\sigma_n := (t_n, t_{n+1}], (n = 1, \cdots, N-$

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$1), Z_N := \{t_n : 1 \leqq n \leqq N-1\}$ (the set of interior grid points). The quantity $h, h := max\{h_n := t_{n+1} - t_n : 0 \leqq n \leqq N-1\}$, is often called the diameter of the grid Π_N. If $h_n \equiv h$ for all $0 \leqq n \leqq N-1$, then the grid Π_N is called a uniform mesh.

The desired approximation to y is the element $u \in S_{m-1}^{(d)}(Z_N, T)$ satisfying

$$u(t) = \int_0^t p(t,s)k(t,s,u(s))\,ds + g(t), \qquad t \in X(N), \tag{1.4}$$

where $X(N) := \bigcup_{n=0}^{N-1} X_n$ with

$$X_n := \{t_{nj} := t_n + c_j h_n : 0 \leqq c_1 < \cdots < c_m \leqq 1\},$$

where $\{c_j\}_{j=1}^m$ are collocation parameters.

The published results on the collocation methods for VIEs (1.1) can be summarized as follows (see [**5**]): if the given functions p, k and g are m times continuously differentiable on their respective domains, it follows that the corresponding solutions have the same degree of smoothness on $I(T)$. This, in turn, implies that the collocation approximation from the above spline space exhibits the (optimal) order of global convergence $\bar{p} = m$. However, if we admit kernels containing a weak singularity of the form, for example, $(t-s)^{-\alpha}$, with $0 < \alpha < 1$, and if we employ uniform mesh sequences, then the global order of convergence of the collocation approximation drops to $\bar{p} = 1 - \alpha$, regardless of how we choose its degree $m - 1$. This is due to the fact that solutions of (1.1) will, in general, be such that $y \in C(I(T))$ but $y'(t) \sim t^{-\alpha}$ as $t \to 0$. In order to restore the optimal order of convergence a graded mesh, $t_i = (i/N)^r T, i = 0, \cdots, N$, with grading exponent of $m/(1-\alpha)$, is used and the resulting collocation approximation exhibits the optimal order of global convergence, *i.e.* $\bar{p} = m$ (see, e.g. [**3**]). However, in using graded meshes we have to start the collocation method on a very small subinterval. This may create serious round-off errors in subsequent calculations, which will restrict applicability of the method.

In this paper we shall investigate the close relationship between the convergence rate and the analytical behavior of the solutions. It will be shown that, for a class of singular kernels, if the exact solutions of VIEs are smooth then the use of uniform meshes yields optimal convergence rates (cf. Abdalkhani [**1**] who considered the kernel of the form $(t-s)^{-\alpha}$). If the exact solution of VIE is not smooth, the independent variable is suitably transformed so that the solution is smooth under this new independent variable. Then collocation methods can be employed to obtain an approximate solution and the numerical methods can be proved to yield optimal convergence rates. This gives a practical numerical approach to solving VIEs with weakly singular kernels.

In order to prove convergence of collocation methods, most authors have used discrete Gronwall inequalities (see, e.g. [**2, 7, 8**]) in their proofs (see, e.g. [**3, 6, 10, 11**]). However, for each function $p(t,s)$ a different Gronwall inequality is required and even with this the analysis of convergence is not always easy to demonstrate. To avoid this difficulty, we write the inequalities involving the error function in continuous form so that we can apply a continuous Gronwall inequality. This provides us with a general tool for investigating the error analysis. Norbury and Stuart [**10**] also regularise the behaviour of the solution by introducing a new independent variable and use a continuous

Gronwall lemma to prove convergence. With their transformation they achieve $O(h^{3/2-\varepsilon})$ convergence, whereas using the transformation of this paper orders of convergence as high as one wishes can be achieved. The paper concludes with a table of numerical results to demonstrate this point.

2. A generalised Gronwall-type inequality

Throughout this paper, C_i, where i is an integer, will denote constants which are independent of h (or, equivalently, the partition Π_N).

DEFINITION 2.1. Let $p_1(t,s) := p(t,s)$, and set

$$p_n(t,s) := \int_s^t p_1(t,\tau)p_{n-1}(\tau,s)\,d\tau, \qquad (t,s)\in S, \quad n \geqq 2, \tag{2.1}$$

where $S := \{(t,s), 0 \leqq s < t \leqq T\}$. The functions $\{p_n, n = 1,2,\cdots\}$ are called *the iterated kernels associated with the given kernel p.*

DEFINITION 2.2. If the function p satisfies

$$(i) \qquad p(t,s) \geqq 0, \quad (t,s)\in S, \tag{2.2}$$

$$(ii) \qquad \int_0^t p(t,s)\,ds \leqq C_1, \quad t \in I(T), \tag{2.3}$$

$$(iii) \qquad p_\nu(t,s) \leqq C_2, \quad (t,s)\in S, \tag{2.4}$$

where ν is a certain integer, then p is said to *satisfy Conditions C*.

THEOREM 2.3. *Let $A \geqq 0$ be a constant, and let the function x satisfy Condition C. The function $x(t)$ is defined as*

$$x(t) = \kappa_n, \qquad t\in[t_n, t_{n+1}), \qquad 0 \leqq n \leqq N-1, \tag{2.5}$$

where t_n is given by (1.3) and $\kappa_n \geqq 0$. If the function x satisfies the integral inequality

$$x(t) \leqq \int_0^t p(t,s)x(s)\,ds + A, \qquad t\in[0,T), \tag{2.6}$$

then it can be bounded by

$$x(t) \leqq C_2 \int_0^t x(s)\,ds + C_3 A, \qquad t\in[0,T).$$

Furthermore, if $h := max\{h_n := t_{n+1} - t_n, 0 \leqq n \leqq N-1\} \leqq C_4/N$, then

$$\kappa := \max\{\kappa_n, 0 \leqq n \leqq N-1\} \leqq C_5 A. \tag{2.7}$$

Proof. Consider

$$x(s) \leqq \int_0^s p(s,\lambda)x(\lambda)\,d\lambda + A, \quad s \in I(T).$$

Multiplying this inequality by $p(t,s)$ and integrating the resulting inequality from 0 to t we obtain

$$\int_0^t p(t,s)x(s)\,ds \leqq \int_0^t p_2(t,s)x(s)\,ds + C_1 A; \tag{2.8}$$

here we have made use of (2.3) and the Dirichlet's formula which states that

$$\int_0^t \int_0^s \Phi(s,\lambda)\,d\lambda\,ds = \int_0^t \int_\lambda^t \Phi(s,\lambda)\,ds\,d\lambda$$

provided the integrals exist. If we replace the first term on the right-hand side of (2.6) using (2.8), we are led to

$$x(t) \leqq \int_0^t p_2(t,s)x(s)\,ds + A(1+C_1).$$

Repeating the above procedures, we can obtain

$$x(t) \leqq \int_0^t p_\nu(t,s)x(s)\,ds + A\sum_{j=0}^{\nu-1} C_1^j.$$

From (2.4) we have

$$x(t) \leqq C_2 \int_0^t x(s)\,ds + C_3 A, \tag{2.9}$$

where $C_3 := \sum_{j=0}^{\nu-1} C_1^j$. Noting that $h \leqq C_4/N$, we obtain from (2.5) and (2.9) that

$$\kappa_n \leqq C_2 C_4 \sum_{i=0}^{n-1} \kappa_i \frac{1}{N} + C_3 A, \qquad 0 \leqq n \leqq N-1.$$

The above inequality is the standard discrete Gronwall inequality which yields (2.7). $\square$

LEMMA 2.4. *If $p(t,s) = \lambda T^{1-\alpha} s^{\lambda-1}(t^\lambda - s^\lambda)^{-\alpha}$, with $T > 0, \lambda \geqq 1, 0 < \alpha < 1$ constants, and $(t,s) \in S_1 := \{(t,s), 0 \leqq s < t \leqq 1\}$, then p satisfies Condition C.*

Proof. It is easy to see that $p(t,s) \geqq 0$ for $(t,s) \in S_1$. Also,

$$\int_0^t p(t,s)\,ds = \frac{T^{1-\alpha}}{1-\alpha} t^{\lambda(1-\alpha)} \leqq \frac{T^{1-\alpha}}{1-\alpha},$$

for $t \in I(1)$. Further, it can be shown that the functions $\{p_n\}$ defined in (2.1) are

$$p_n(t,s) = \frac{(\Gamma(1-\alpha))^n}{\Gamma(n(1-\alpha))} \lambda T^{n(1-\alpha)} s^{\lambda-1}(t^\lambda - s^\lambda)^{(n-1)(1-\alpha)-\alpha},$$

where Γ is the Gamma function. Let ν be the smallest integer such that $(\nu-1)(1-\alpha) \geqq \alpha$. Since $\lambda \geqq 1$, we have $p_\nu \in C(S_1)$ and therefore p_ν is bounded in S_1. $\square$

3. Convergence of collocation methods.

Throughout this paper, w write $E = \varepsilon(h)$ as shorthand for the inequality $|E| \leqq Ch^\delta$ with C and δ positive constants.

DEFINITION 3.1. If a function p satisfies Condition C and

$$(i) \qquad \int_{t_n}^{t_{nj}} p(t_{nj}, s)\, ds = \varepsilon(h), \tag{3.1}$$

$$(ii) \qquad \int_0^{t_n} \Big| p(t_{nj}, s) - p(t, s) \Big|\, ds = \varepsilon(h), \quad t \in [t_n, t_{n+1}), \tag{3.2}$$

where $t_{nj} \in X_n, 0 \leqq n \leqq N-1$, then p is said *to satisfy Condition D.*

THEOREM 3.2. *Let the function p in (1.1) satisfy Condition D, and* $H(t,s,z) := k_z(t,s,z)$ *satisfy*

$$|H(t,s,z)| \leqq C_6, \qquad (t,s) \in S, \quad -\infty < z < \infty. \tag{3.3}$$

If the solution y of (1.1) belongs to $C^m(I(T))$ with $m \geqq 1$, then for a uniform mesh sequence and for any choice of the collocation parameters $\{c_j\}$ with $0 \leqq c_1 < \cdots < c_m \leqq 1$, the error $e(t) := y(t) - u(t)$ satisfies

$$\|e\|_\infty := \max\{|e(t)|, \quad t \in I(T)\} = O(h^m), \quad h \ll 1, \tag{3.4}$$

where u is the solution of the collocation equation (1.4) and h $(= T/N)$ is the step size of the uniform mesh sequence.

REMARK 3.3. For the function k it is only necessary to require Lipschitz continuity with respect to the third variable. However, this complicates the argument somewhat, and for reasons of clarity we consider condition (3.3).

Proof of Theorem 3.2. Set $t = t_{nj}$ in (1.1), and subtract the collocation equation (1.4) from the resulting equation. Denoting by $e_n := y - u_n$ the restriction of the collocation error to the subinterval σ_n, we obtain

$$\begin{aligned} e_n(t_{nj}) = h &\int_0^{c_j} p(t_{nj}, t_n + \nu h) H(t_{nj}, t_n + \nu h, \xi_n) e_n(t_n + \nu h)\, d\nu \\ &+ \sum_{i=0}^{n-1} h \int_0^1 p(t_{nj}, t_i + \nu h) H(t_{nj}, t_i + \nu h, \xi_i) e_i(t_i + \nu h)\, d\nu, \end{aligned} \tag{3.5}$$

where $\xi_i \in (\min(y, u_i), \max(y, u_i))$. Here we have made use of the mean value theorem applied to the third variable of the function k. For $\nu \in (0, 1]$ (if $n = 0, \nu \in [0, 1]$), we follow Brunner [**3**] and write

$$e_n(t_n + \nu h) = \sum_{l=1}^m \beta_{nl} \nu^{l-1} + h^m R_n(\nu), \quad 0 \leqq n \leqq N-1, \tag{3.6}$$

where β_{nl} are constants, and

$$R_n(\nu) = y^{(m)}(t_n + \theta_n \nu h) \nu^m / m! \qquad (0 < \theta_n < 1).$$

Combining (3.5) and (3.6) results in a recurrence relation for the vectors $\beta_n := (\beta_{n1}, \cdots, \beta_{nm})^T$, namely,

$$\begin{aligned}
&\sum_{l=1}^{m} \beta_{nl} \left\{ c_j^{l-1} - h \int_0^{c_j} p(t_{nj}, t_n + \nu h) H(t_{nj}, t_n + \nu h, \xi_n) \nu^{l-1}\, d\nu \right\} \\
&= \sum_{i=0}^{n-1} h \sum_{l=1}^{m} \beta_{il} \int_0^1 p(t_{nj}, t_i + \nu h) H(t_{nj}, t_i + \nu h, \xi_i) \nu^{l-1}\, d\nu + q_{nj}, \qquad (3.7)
\end{aligned}$$

where the remainder terms are given by

$$\begin{aligned}
q_{nj} := &-h^m R_n(c_j) + h \int_0^{c_j} p(t_{nj}, t_n + \nu h) H(t_{nj}, t_n + \nu h, \xi_n)(h^m R_n(\nu))\, d\nu \\
&+ \sum_{i=0}^{n-1} h \int_0^1 p(t_{nj}, t_i + \nu h) H(t_{nj}, t_i + \nu h, \xi_i)(h^m R_i(\nu))\, d\nu. \qquad (3.8)
\end{aligned}$$

For $0 \leqq i \leqq n-1$, define the $m \times m$ matrices $\mathbf{D}_{ni}$ by

$$\mathbf{D}_{ni} := \left(h \int_0^1 p(t_{nj}, t_i + \nu h) H(t_{nj}, t_i + \nu h, \xi_i) \nu^{l-1}\, d\nu \right),$$

(where both j and l vary between 1 and m). Moreover, let

$$\mathbf{D}_{nm} := \left(h \int_0^{c_j} p(t_{nj}, t_n + \nu h) H(t_{nj}, t_n + \nu h, \xi_n) \nu^{l-1}\, d\nu \right),$$

and let $\mathbf{V} := (c_j^{l-1})$ denote the Vandermonde matrix of order m associated with the collocation parameters $\{c_j\}$. The recurrence relation (3.7) can thus be written as

$$(\mathbf{V} - \mathbf{D}_{nn})\beta_n = \sum_{i=0}^{n-1} \mathbf{D}_{ni}\beta_i + \mathbf{q}_n, \quad 0 \leqq n \leqq N-1, \qquad (3.9)$$

where $\mathbf{q}_n := (q_{n1}, \cdots, q_{nm})^T$ is the vector whose components are defined by (3.8). Since p satisfies (3.1) and H satisfies (3.3), we have

$$\begin{aligned}
&\left| h \int_0^{c_j} p(t_{nj}, t_n + \nu h) H(t_{nj}, t_n + \nu h, \xi_n) \nu^{l-1}\, d\nu \right| \\
&\leqq C_6 h \int_0^{c_j} p(t_{nj}, t_n + \nu h)\, d\nu = C_6 \int_{t_n}^{t_{nj}} p(t_{nj}, s)\, ds \leqq \varepsilon(h).
\end{aligned}$$

Hence the matrix $\mathbf{V} - \mathbf{D}_{nn}$ possesses a uniformly bounded inverse for sufficiently small h. Thus there exists a finite constant C_7, independent of h and N, such that

$$\|(\mathbf{V} - \mathbf{D}_{nn})^{-1}\|_1 \leqq C_7, \qquad 0 \leqq n \leqq N-1. \qquad (3.10)$$

Further

$$\|\mathbf{D}_{ni}\|_1 \leqq \sum_{j=1}^{m} C_6 h \int_0^1 p(t_{nj}, t_i + \nu h)\, d\nu = C_6 \sum_{j=1}^{m} \int_{t_i}^{t_{i+1}} p(t_{nj}, s)\, ds. \qquad (3.11)$$

Then from (3.9), (3.10) and (3.11) we obtain that

$$\|\beta_n\|_1 \leqq C_8 \sum_{i=0}^{n-1} \sum_{j=1}^{m} \int_{t_i}^{t_{i+1}} p(t_{nj}, s)\, ds \|\beta_i\|_1 + C_7 \|\mathbf{q}_n\|_1, \quad 0 \leqq n \leqq N-1, \tag{3.12}$$

where $C_8 := C_6 C_7$. Let $x(t) = \|\beta_n\|_1$, $t \in \sigma_n, 0 \leqq n \leqq N-1$. For any $t \in \sigma_n$, we have from (3.12) that

$$\begin{aligned} x(t) &\leqq C_8 \sum_{j=1}^{m} \int_0^{t_n} p(t_{nj}, s) x(s)\, ds + C_7 \|\mathbf{q}_n\|_1 \\ &= C_8 \sum_{j=1}^{m} \int_0^{t_n} p(t, s) x(s)\, ds + C_7 \|\mathbf{q}_n\|_1 + C_8 \sum_{j=1}^{m} \int_0^{t_n} \Big[p(t_{nj}, s) - p(t, s) \Big] x(s)\, ds \\ &\leqq m C_8 \int_0^{t_n} p(t, s) x(s)\, ds + C_7 \|\mathbf{q}_n\|_1 + C_8 \beta \sum_{j=1}^{m} \int_0^{t_n} \Big| p(t_{nj}, s) - p(t, s) \Big|\, ds, \end{aligned} \tag{3.13}$$

where $\beta := \max\{\|\beta_n\|_1, 0 \leqq n \leqq N-1\}$. The last term of (3.13) can be bounded by

$$C_8 \beta \sum_{j=1}^{m} \int_0^{t_n} \Big| p(t_{nj}, s) - p(t, s) \Big|\, ds = m C_8 \beta \varepsilon(h), \tag{3.14}$$

where we have made use of (3.2). Since $t \geqq t_n$ and the functions x and p are non-negative, we obtain from (3.13) and (3.14) that

$$x(t) \leqq m C_8 \int_0^t p(t, s) x(s)\, ds + C_7 \|\mathbf{q}_n\|_1 + m C_8 \beta \varepsilon(h). \tag{3.15}$$

Since $y \in C^m(I(T))$, and the functions p and H satisfy (2.3) and (3.3) respectively, we can easily show that

$$\|\mathbf{q}_n\|_1 \le C_9 (1 + C_1 C_6) h^m. \tag{3.16}$$

Applying Theorem 2.3 to the inequalities (3.15) and (3.16), we obtain that

$$\beta \leqq C_{10} \Big(h^m + \varepsilon(h) \beta \Big).$$

The above inequality gives

$$\Big(1 - C_{10} \varepsilon(h) \Big) \beta \leqq C_{10} h^m,$$

which is equivalent to (3.4) provided that h is sufficiently small. $\square$

4. An application

In this section we consider the case when $p(t,s) = (t-s)^{-\frac{1}{2}}$. The results in this section can be easily extended to the case when $p(t,s) = (t-s)^{-\alpha}$ (see [**12**]). If $p(t,s) = (t-s)^{-\frac{1}{2}}$, and if k and g have continuous derivatives of order m, then there exists a function

$Z = Z(t, v)$ possessing continuous derivatives of order m, such that the solution of (1.1) can be written as $y(t) = Z(t, t^{1-\frac{1}{2}})$, (see e.g. [**9**]). This implies that near $t = 0$ its first derivative behaves like $y'(t) \sim t^{-\frac{1}{2}}$. If we apply transformations

$$t := \tilde{t}^2 T, \quad Y(\tilde{t}) := y(t), \tag{4.1}$$

with $\tilde{t} \in I(1)$, then $Y(\tilde{t}) \in C^m(I(1))$. Applying the transformations (4.1) to (1.1), we obtain that

$$Y(\tilde{t}) = \int_0^{\tilde{t}} q(\tilde{t}, s) K(\tilde{t}, s, Y(s))\, ds + G(\tilde{t}), \quad \tilde{t} \in I(1), \tag{4.2}$$

where

$$q(\tilde{t}, s) := 2\sqrt{T} s(\tilde{t}^2 - s^2)^{-\frac{1}{2}}, \quad K(\tilde{t}, s, z) := k(\tilde{t}^2 T, s^2 T, z), \quad G(\tilde{t}) := g(\tilde{t}^2 T). \tag{4.3}$$

For the uniform mesh $\tilde{\Pi}_N$ of the interval $I(1)$

$$\tilde{\Pi}_N : v_n := nh, \quad h = 1/N, \quad 0 \leqq n \leqq N,$$

the desired approximation to Y is the element $U \in S_{m-1}^{(d)}(\tilde{Z}_N, 1)$ satisfying

$$U(\tilde{t}) = \int_0^{\tilde{t}} q(\tilde{t}, s) K(\tilde{t}, s, U(s))\, ds + G(\tilde{t}), \quad \tilde{t} \in \tilde{X}(N), \tag{4.4}$$

where

$$\tilde{Z}_N := \{v_n := nh, h = 1/N, 0 \leqq n \leqq N - 1\}$$

and

$$\tilde{X}(N) := \{v_{nj} := nh + c_j h : 0 \leqq c_1 < \cdots < c_m \leqq 1, \quad 0 \leqq n \leqq N - 1\}.$$

THEOREM 4.1. *Let k and g in (1.1) be m-times continuously differentiable on their respective domains. If $k_z(t, s, z)$ is bounded for $(t, s) \in S$ and $z \in (-\infty, \infty)$, then for uniform mesh sequence and for any choice of the collocation parameters $\{c_j\}$ with $0 \leqq c_1 < \cdots < c_m \leqq 1$, the error function $E(\tilde{t}) := Y(\tilde{t}) - U(\tilde{t})$ satisfies*

$$\|E\|_\infty := \max\Big\{|E(\tilde{t})|,\ \tilde{t} \in I(1)\Big\} = O(h^m), \tag{4.5}$$

when $h = 1/N$ is sufficiently small. Here, Y and U are the solutions of (4.2) and (4.4), respectively.

Proof. Since $Y \in C^m(I(1))$, we need only to show that q, which is defined by (4.3), satisfy Condition D. First of all, we have

$$\int_{nh}^{nh+c_j h} q(nh + c_j h, s)\, ds = 2\sqrt{T}\Big((nh + c_j h)^2 - (nh)^2\Big)^{\frac{1}{2}} \leqq 2\sqrt{2Th} = \varepsilon(h), \tag{4.6}$$

where we have made use of the mean-value theorem and the fact that $(n + 1)h \leqq 1$. Noting that $q(t, s) \geqq q(nh + c_j h, s)$ for $t \in [nh, nh + c_j h)$, we have

$$\begin{aligned} \int_0^{nh} \Big| q(nh + c_j h, s) - q(t, s) \Big|\, ds &= \int_0^{nh} \Big[q(t, s) - q(nh + c_j h, s) \Big]\, ds \\ &= 2\sqrt{T}[t - (t^2 - (nh)^2)^{\frac{1}{2}} + ((nh + c_j h)^2 - (nh)^2)^{\frac{1}{2}} - (nh + c_j h)] \\ &\leqq 2\sqrt{T}((nh + c_j h)^2 - (nh)^2)^{\frac{1}{2}} = \varepsilon(h). \end{aligned} \tag{4.7}$$

In a similar way, we can prove that (4.7) is also true for $t \in [nh+c_jh, (n+1)h)$. Combining this with the result of Lemma 2.4, we observe that the function q satisfies Condition D. An application of Theorem 3.2 then yields (4.5) which completes the proof. $\square$

As mentioned in Section 1, although a graded mesh with a suitable grading exponent can give the optimal order of global convergence, the initial step size becomes very small as N is incresed, thus presenting a potential source of rounding errors in subsequent calculations. Clearly, the transformations (4.1) can minimise this difficulty and lead to an accurate collocation approximation U. Moreover, from (4.1) and (4.5) we obtain that

$$\|y - \bar{u}\|_\infty = O(h^m),$$

where y is the exact solution of (1.1) with $p(t,s) = (t-s)^{-\frac{1}{2}}$, and $\bar{u} := U(\sqrt{t/T})$.

5. Numerical experiment

In order to illustrate the theoretical results of Section 4, we consider the linear equation

$$y(t) = \int_0^t (t-s)^{-\frac{1}{2}} k(t,s) y(s)\, ds + g(t), \quad t \in I(1). \tag{5.1}$$

The numerical procedure described in this section can be extended to the nonlinear equation (1.1) following [**3**]. After applying the transformation (4.1) with $T = 1$, we obtain

$$Y(\tilde{t}) = \int_0^{\tilde{t}} q(\tilde{t},s) K(\tilde{t},s) Y(s)\, ds + G(\tilde{t}), \quad \tilde{t} \in I(1), \tag{5.2}$$

where $q(\tilde{t},s) = 2s(\tilde{t}^2 - s^2)^{-\frac{1}{2}}$, $K(\tilde{t},s) = k(\tilde{t}^2, s^2)$ and $G(\tilde{t}) = g(\tilde{t}^2)$. The collocation equation for the approximation U to Y which belongs to $S_{m-1}^{(d)}(\tilde{Z}_N, 1)$ has the form

$$U(\tilde{t}) = \int_0^{\tilde{t}} q(\tilde{t},s) K(\tilde{t},s) U(s)\, ds + G(\tilde{t}), \quad \tilde{t} \in \tilde{X}(N). \tag{5.3}$$

The product trapezoidal method

For the choice of $m = 2$, $c_1 = 0$ and $c_2 = 1$, we generate a continuous piecewise linear approximation $U \in S_1^{(0)}(\tilde{Z}_N, 1)$. The resulting collocation equation (5.3) can be written as

$$U(v_n) = \sum_{j=0}^{n-1} \int_{v_j}^{v_j+1} q(v_n,s) K(v_n,s) U(s)\, ds + G(v_n), \quad 1 \leqq n \leqq N, \tag{5.4}$$

with starting value $U(0) = Y(0)$. In the above

$$U(s) = U_j l_{j1}(s) + U_{j+1} l_{j2}(s), \quad s \in [v_j, v_{j+1}], \tag{5.5}$$

with $U_j := U(v_j)$ and where l_{j1} and l_{j2} are the linear Lagrange polynomials

$$l_{j1}(s) := (v_{j+1} - s)/h, \quad l_{j2}(s) := (s - v_j)/h.$$

Approximating the integral in (5.4) by product integration formulae, we obtain the following discretised version

$$(1-\omega^{(2)}_{n,n-1})U_n := \sum_{j=0}^{n-2}(\omega^{(1)}_{nj}U_j + \omega^{(2)}_{nj}U_{j+1}) + G(v_n), \tag{5.6}$$

for $1 \leqq n \leqq N$, and with

$$\omega^{(1)}_{nj} := K(v_n, v_j)\int_{v_j}^{v_{j+1}} q(v_n, s) l_{j1}(s)\, ds, \quad 0 \leqq j \leqq n-1, \tag{5.7}$$

$$\omega^{(2)}_{nj} := K(v_n, v_{j+1})\int_{v_j}^{v_{j+1}} q(v_n, s) l_{j2}(s)\, ds, \quad 0 \leqq j \leqq n-1. \tag{5.8}$$

The integrals in the above formulae can be easily computed analytically, e.g. the integral in (5.7) can be expressed as

$$2\sqrt{v_n^2 - v_j^2} - \frac{v_n^2}{h}[\, \arcsin(v_{j+1}/v_n) - \arcsin(v_j/v_n)\,] - \frac{1}{h}[\, v_{j+1}\sqrt{v_n^2 - v_{j+1}^2} - v_j\sqrt{v_n^2 - v_j^2}\,].$$

The product Simpson's method

Here the choice $m = 3$, $c_1 = 0$, $c_2 = \frac{1}{2}$ and $c_3 = 1$ implies that the approximation $U \in S_2^{(0}(\tilde{Z}_n, 1)$ and we have

$$U(s) = U_{j1}L_{j1}(s) + U_{j2}L_{j2}(s) + U_{j3}L_{j3}(s), \quad s \in [v_j, v_{j+1}],$$

with $U_{ji} := U(v_{ji})$, $i = 1, 2, 3$, and where the L_{ji} are the Lagrange polynomials associated with v_{ji}. The collocation equation (5.3) has the form

$$\begin{aligned} U(v_{ni}) \;&=\; \sum_{j=0}^{n-1}\int_{v_j}^{v_{j+1}} q(v_{ni}, s)K(v_{ni}, s)U(s)\, ds \\ &\quad + \int_{v_n}^{v_{ni}} q(v_{ni}, s)K(v_{ni}, s)U(s)\, ds + G(v_{ni}), \end{aligned} \tag{5.9}$$

for $i = 2, 3$ and $0 \leqq n \leqq N-1$. To approximate the integrals in (5.9), we use the following quadrature formulae:

$$\int_{v_j}^{v_{j+1}} q(v_{ni}, s)K(v_{ni}, s)U(s)\, ds = \gamma^{(1)}_{nji}U_{j1} + \gamma^{(2)}_{nji}U_{j2} + \gamma^{(3)}_{nji}U_{j3}, \tag{5.10}$$

with

$$\gamma^{(l)}_{nji} := K(v_{ni}, v_{jl})\int_{v_j}^{v_{j+1}} q(v_{ni}, s)L_{jl}(s)\, ds, \quad l = 1, 2, 3. \tag{5.11}$$

The quadrature formulae for the last integral in (5.9) can be obtained in a similar way as described in [**3**]. Since $L_{jl}(s)$ is a polynomial of degree 2, the integrals in (5.11) can be computed analytically if and only if we can find $\int_{v_j}^{v_{j+1}} s^l q(v,s)\,ds$, $l = 0, 1$ and 2 exactly. It can be shown that

$$\int_{v_j}^{v_{j+1}} s^l q(v,s)\,ds = 2v^{l+1} \int_{\arcsin(v_j/v)}^{\arcsin(v_{j+1}/v)} \sin^{l+1}\theta\,d\theta. \tag{5.12}$$

It is easy to see that the right-hand side of (5.12) can be evaluated analytically. Using the above formulae and with the starting value $U_0 = Y(0)$, we can obtain, for each $0 \leqq n \leqq N-1$, a system of two equations in the unknowns U_{n2} and $U_{n3}(= U_{n+1,1})$. These equations can be solved by inverting a 2×2 matrix.

Table 5.1: Errors and convergence rates.

N	Trapezoidal		Simpson		Norbury and Studart	
	$\Vert e\Vert_\infty$	rate	$\Vert e\Vert_\infty$	rate	$\Vert e\Vert_\infty$	rate
10	6.22D-3		1.60D-4		6.05D-3	
20	1.74D-3	1.84	2.24D-5	2.84	2.21D-3	1.45
40	4.26D-4	1.91	2.96D-6	2.92	8.09D-4	1.45
80	1.19D-4	1.96	3.81D-7	2.96	2.95D-4	1.46

The product trapezoidal method and the product Simpson's method were implemented as Fortran programs in double precision for the following example.

Example 5.1. [**9**] Consider

$$y(t) = -\int_0^t (t-s)^{-\frac{1}{2}} y(s)\,ds + 1, \quad 0 \leqq t \leqq 1. \tag{5.13}$$

The above equation has the exact solution $y(t) = \exp(\pi t)\,\mathrm{erfc}\,(\sqrt{\pi t})$. In Table 5.1 we list the errors, $\Vert e\Vert_\infty$, and computed rates of convergence which indicate convergence of order two for the trapezoidal method and three for the Simpson method. This confirms the theoretical results in Section 4. We have also applied to the above example the algorithm by Norbury and Stuart ([**10**]) which is based on a trapezium approximation. The errors in Table 5.1 confirm their predicted convergence order $p \approx 1.5$, and this is weaker than the results of this paper.

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