# PETER D. LAX
# ONE EXAMPLE FROM HIS CONTRIBUTIONS TO MATHEMATICS

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## 1. INTRODUCTION

Peter D. Lax has made seminal contributions to several key areas of mathematics. His contributions are part of a long tradition where the interaction between mathematics and physics is at the core. Physics offers challenging problems that require intuition to solve. Mathematics can reveal deep inner strucures and properties, and rigorous proofs provide solid foundations for our knowledge. John von Neumann, who had considerable influence on Lax, concluded in 1945 that[1] "really efficient high-speed computing devices may, in the field of non-linear partial differential equations as well as in many other fields which are now difficult or entirely denied of access, provide us with those heuristic hints which are needed in all parts of mathematics for genuine progress." Lax stated in 1986 that[2] "[a]pplied and pure mathematics are more closely bound together today than any other time in the last 70 years". It is in this spirit that Lax has worked.

Peter D. Lax

Here we will focus on one important area of mathematics were Peter Lax has made outstanding contributions that continue to dominate the field. We emphasis the applied aspects with wide ranging consequences for our modern society, thereby underplaying their intrinsic mathematical beauty. Unfortunately we will not discuss his fundamental contributions to classical analysis and scattering theory, in particular *Lax–Phillips scattering theory*, nor his contributions to the theory of solitons, where he introduced the fundamental *Lax pair*, which has had extensive consequences in mathematics, physics and technology.

The topic we will discuss is the theory of shock waves. Shock waves appear in many phenomena in everyday life. Most easily explained are shock waves coming from airplanes moving at supersonic speed, or from explosions, but shocks also appear in phenomena involving much smaller velocities. Of particular interest is the flow of hydrocarbons in a porous

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[1] *Collected works of John v. Neumann*, vol. V, 1963, p. 1–32.

[2] Mathematics and its applications, *The Mathematical Intelligenzer* **8** (1986) 14–17.

medium, or to put it more concretely, the flow of oil in an oil reservoir. It is well-known that oil and water do not mix, and the interface between regions with oil and regions with water form what is mathematically defined as a shock. The dynamics of the shocks are vital in the exploitation of hydrocarbons from petroleum reservoirs. Even in everyday phenomena like traffic jam on heavily congested roads, we experience shock waves when there is an accumulation of cars. The shocks do not come from collisions of cars, but rather from a rapid change in the density of cars.

A more extensive discussion of several aspects of Peter Lax's contributions to mathematics can be found in [1]. An interview with him appeared in [2], and the full range of his contributions can be studied in his recently published selected works [3].

We must first explain what a differential equation is.

## 2. WHAT IS A DIFFERENTIAL EQUATION?

In order to discuss differential equations, we first have to introduce the derivative. Consider the situation when you are driving your car. On the odometer, you can measure the distance from your starting point, and knowing that, your position is determined. The distance you cover per unit of time is called the velocity, and that, of course, is what is displayed on the speedometer. Mathematically, the velocity is nothing but the derivative of the position. To put this in mathematical terms, we let $x$ denote the position of the car, measured along the road from some starting point. It depends on time, $t$, so we write that $x = x(t)$. The velocity, which we denote by $v$ and which depends on time, $v = v(t)$, is the change of position for a small time interval, and mathematically we call that the derivative[3] of $x$, and write $x'(t)$. Thus $v(t) = x'(t)$.

If a passenger in the car at each instant of time notes down the velocity, it should be possible to compute the position of the car at each point in time if we know the time and place that the trip started. To put it more precisely, if we know the starting point $x_0$ (and synchronize our clocks so that we start at time $t = 0$), thus $x(0) = x_0$, and we know $v(t)$ for all $t$, we should be able to compute the position $x$ as a function of time, that is, determine $x = x(t)$. To solve this problem we have to solve a differential equation, namely $x'(t) = v(t)$.

Differential equations are nothing but equations that involve derivatives. You may think that we are doing a lot out of a small problem. However, it turns out that all the fundamental laws of nature can be expressed as differential equations, as the following list displays

- Gravitation (Newton's law),

[3] To make it more precise, if you advance from position $x(t)$ at time $t$ to position $x(t+s)$ during the time period $s$, the velocity at time $t$ is approximately $(x(t+s)-x(t))/s$, and the approximation gets better the smaller time interval $s$ you use. Mathematically, the velocity equals the limit of $(x(t+s)-x(t))/s$ as $s$ tends to zero.

- Quantum mechanics (The Schrödinger equation),
- Electromagnetism (Maxwell's equations),
- Relativity (Einstein's equations),
- The motion of gases and fluids (The Navier–Stokes's equations).

The motion of planets, computers, electric light, the working of GPS (Global Positioning System), and the changing weather can all be described by differential equations.

Let us proceed to a more complicated example than the position and velocity of cars. Consider the heat in the room where you are sitting. At each point $(x, y, z)$ in space and time $t$ we let $T = T(x, y, z, t)$ denote the temperature. Assuming that walls are isolating and that no ovens are on in the room, we can derive that the temperature distribution is determined by the so-called heat equation, which reads

$$T_t = T_{xx} + T_{yy} + T_{zz}.$$

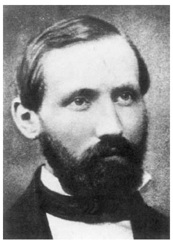
Here $T_t$ means the derivative of the temperature with respect to the variable $t$, while $T_{xx}$ denotes the derivative of the derivative, both with respect to the space variable $x$, and similarly for the remaining terms. Even simple problems give rise to difficult differential equations! Assuming that we know the initial temperature, that is, we know $T = T(x, y, z, t)$ for $t = 0$, our intuition tells us that the temperature should be determined at all later times. The mathematical challenge is to prove this assertion and describe a method to compute the actual temperature. These are the kinds of questions that Peter Lax has addressed in his research on differential equations.

Unfortunately, differential equations do not normally possess solutions that are expressed by formulas, and thus we need to develop methods for computing the solution numerically. With high speed computers, we can determine an approximate, or numerical, solution. We emphasize that by necessity this will be an approximate solution, and we will need to know if, and to what extent, we can trust the approximate answer. In a situation where we do not know the exact answer, or where we cannot know if we were right before it is too late (for instance, the weather forecast for tomorrow), it is absolutely necessary to know to what extent our simulations are reliable. Peter Lax has made fundamental contributions to our understanding of these issues.

There is no unified mathematical theory that covers all, or most, differential equations. Different classes of differential equations require rather different methods, but even at this very general level, Lax has contributed two highly useful results that are described in all books in the area. The *Lax–Milgram theorem* states that certain differential equations have a unique solution. The *Lax equivalence principle* states that any consistent numerical method for linear problems is stable if and only it is convergent.

It is appropriate here to digress briefly on the interaction between mathematics and computers. Peter Lax has always been a strong proponent of the importance of computers to mathematics and vice versa, saying that[4] "[High speed computers'] impact on mathematics, both applied and pure, is comparable to the role of telescopes in astronomy and microscopes in biology". The logical construction of computers and their operating systems are mathematical by nature, but computers also serve as laboratories for mathematicians, where you can test your ideas. New mathematical relations can be discovered, and your hypotheses and assumptions can be disproved or made more likely by applying computers. Lax has given the example of the great mathematician G. D. Birkhoff who spent a lifetime trying to prove the ergodic hypothesis. If Birkhoff had had access to a computer and had tested the hypothesis on it, he would have seen that it cannot be true in general. On a more technical level, problems of modern technology like the simulation of systems as complex as airplanes, oil rigs, or the weather not only require very powerful computers, but also the development of new and better mathematical algorithms for their solution. It is a fact that in broad terms, the development of high speed computers (hardware) and the development of new numerical techniques (software) have contributed equally to the total performance we observe in simulations. Peter Lax himself has made penetrating contributions to the development of new mathematical methods that have enabled us to understand and compute important phenomena.

## 3. SHOCK WAVES


B. Riemann

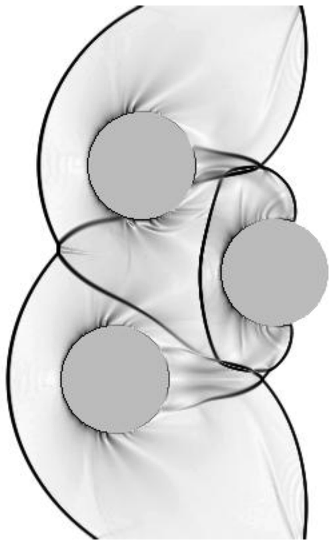
In 1859 the brilliant German mathematician Berhard Riemann (1826–66) considered the following problem: If you have two gases at different pressures in a cylinder separated by a thin membrane, what happens if you remove the membrane? This problem has later been called the Riemann problem, and it turns out to be a very complicated question. The behavior of gases is well modeled by the Euler equations, which read[5]

$$\begin{aligned} \rho_t + (\rho v)_x &= 0, \\ (\rho v)_t + (\rho v^2 + P)_x &= 0, \\ E_t + (v(E+P))_x &= 0, \\ P &= P(\rho), \end{aligned}$$

where $p$, $v$, $P$, and $E$ denote the density, velocity, pressure, and energy of the gas, respectively. This is truely an intricate system of equations that remains unsolved in the general case to this day..

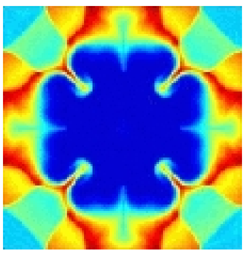
[4]The flowering of applied mathematics in America, *SIAM Review* **31** (1989) 533–541.

[5]Riemann studied the simpler problem where the third equation, the one for the energy, is ignored.



Flow of gas past three cylinders.

The Euler equations constitute a special case of a class of differential equations called hyperbolic conservation laws. The solutions of these equations are very complicated as the illustrations show. These equations are fundamental in several areas of applied science, for they express that a quantity is preserved. Examples abound because we have conservation of mass, momentum, and energy in isolated systems. In addition to the motion of gases, applications include the flow of oil in a petroleum reservoir. A less obvious example is the dynamics of cars on a highly congested road without exits or entries; here the conserved quantity is the number of cars.



The pressure of a gas exploding in a box.

The core of the problem with hyperbolic conservation laws, regardless of whether they describe traffic flow or the flow of oil in a petroleum reservoir, is that the solution develops singularities, or discontinuities, called shocks. Shocks correspond to very rapid transitions in density or pressure. Numerical methods have difficulty resolving these shocks, and the mathematical properties are very complicated. The mathematical models allow for more than one solution, and the selection principle, which goes under the name of entropy condition, for determining the one true physical solution is very complicated. Indeed, at this point Riemann erred and selected the wrong solution, tacitly assuming that the entropy is preserved. The velocity of the shock was determined by the Scottish engineer, Rankine, and the French mathematician, Hugoniot, but it was left to Peter Lax in 1957 to come up with a simple criterion, now called the *Lax entropy condition*, that selects the true physical solution for general systems of hyperbolic conservation laws. The admissible shocks are called *Lax shocks*. The solution of the Riemann problem is now called the *Lax theorem*, and it is a cornerstone in the theory of hyperbolic conservation laws. His solution has stimulated extensive further research into different entropy conditions applicable to other systems. In particular, the fundamental existence result for the general initial-value problem posed by Glimm, uses the Lax theorem as a building block.

Once we have decided upon a selection principle, we still have to compute the solution. Here, Peter Lax has introduced two of the standard numerical schemes for solving hyperbolic conservation laws, namely the so-called the *Lax–Friedrichs scheme* and the *Lax–Wendroff scheme*. These schemes serve as benchmark tests for other numerical techniques and have served as a starting point for theoretical analysis. Indeed, the Lax–Friedrichs scheme was used by the Russian mathematician Oleĭnik in her constructive proof of the existence and uniqueness of solutions of the inviscid Burgers equation. Another highly useful result is the *Lax–Wendroff theorem*, which states the following: If a numerical scheme for a nonlinear hyperbolic conservation law converges to a limit, then we know that the limit at least is a solution of the equation.

Peter Lax's results in the theory of hyperbolic conservations laws are groundbreaking. They have resolved old problems, and have stimulated extensive new research in the field, and still are at the core of the field.

**Epilogue.** Lax considers himself both a pure and an applied mathematician. His advice to young mathematicians is summarized in[6] "I heartily recommend that all young mathematicians try their skill in some branch of applied mathematics. It is a gold mine of deep problems whose solutions await conceptual as well as technical breakthroughs. It displays an enormous variety, to suit every style; it gives mathematicians a chance to be part of the larger scientific and technological enterprise. Good hunting!"

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