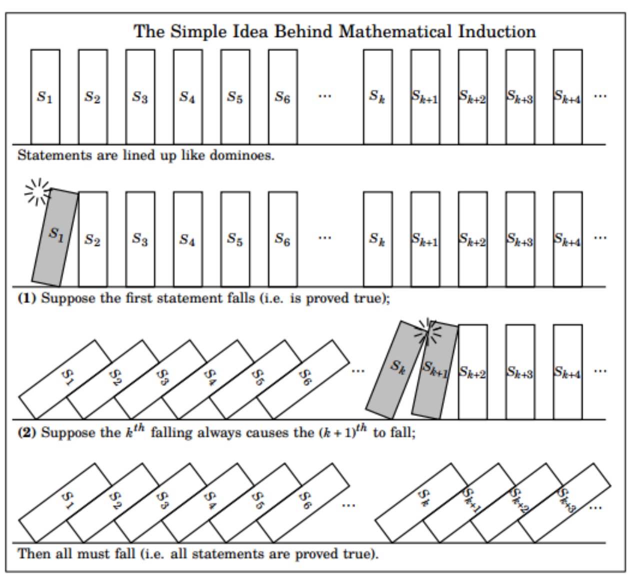
Chapter 1

Mathematical Induction

Mathematical Induction is one simple yet powerful and handy tool to tackle mathematical problems. There are a lot of mathematical theorems that you rely on in your everyday life, which may have been proved using induction, only to later find their way into engineering, and ultimately into the products that you use and on which your very life may depend. Moreover, even if you are not a mathematician yet, but, say, a software developer, engineer, physicist or, for that matter, statistician working in chemistry or biology, you may come across problems as part of your daily work were being able to find/prove a solution using induction can greatly simplify things.

1.1 Dominoes

An analogy of the principle of mathematical induction is the game of dominoes. Suppose the dominoes are lined up properly, so that when one falls, the successive one will also fall. Now by pushing the first domino, the second will fall; when the second falls, the third will fall; and so on. We can see that all dominoes will ultimately fall.



The key point is only two steps:

1. the first domino falls;
2. when a domino falls, the next domino falls.

That is how Mathematical Induction works.

1.2 Principle of Mathematical Induction

Suppose a statement, let us say S, is bound to be verified for each natural number from 1 and up. One says I can show that S is true for 1, easy enough, but thats only one step among infinitely many. But wait, I can show S is true for 2 now also, by making use of the fact that S is true for 1. And now I can show that S is true for 3, using the fact that its true for 1 and 2. Well, after few attempts, you will find out that its just one of those infinite steps taken. To avoid the tedious steps, we shall introduce *Mathematical Induction* in solving these problems, which the inductive proof involves two stages:

1. **The Base Case**: Prove the desired result for number 1.
2. **The Inductive Step**: Prove that if the result is true for any k, then it is also true for the number $k + 1$.

The inductive step is proved by first assuming that the result is true for some k, and then using this assumption to show that it is also true for $k + 1$. It is like when playing dominos. When you push the first domino, it falls, then it knocks the second domino and it falls as well, then at the end, all dominos fall. We may take a look at an example.

Example 1.1. Prove that $1+2+\cdots+n = \dfrac{n(n+1)}{2}$ (1.1) for all positive integers n. One can check whether this holds for the first few sums:

$$\begin{aligned} 1 &= \frac{1(1+1)}{2} = 1 \\ 1+2 &= \frac{2(1+2)}{2} = 3 \\ 1+2+3 &= \frac{3(1+3)}{2} = 6 \\ &\vdots \end{aligned}$$

To be frank, mathematics is a tool to make life easier, not harder. When something tedious is encountered, we shall think of an alternative to solve the problem. In this example, as it is bound to finish the proof by showing infinitely many steps, we may try Mathematical Induction. For $n = 1$,

$$L.H.S = 1 = \frac{1(1+1)}{2} = 1 = R.H.S.$$

Assume it is true for some integer k, (similar for some given domino), i.e.,

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Then, we want to show the next falls, for $n = k + 1$,

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \qquad \text{(By assumption)} \\ &= (k+1)\left(\frac{k}{2} + 1\right) \\ &= (k+1)\left(\frac{k+2}{2}\right) \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

By Mathematical Induction, the statement (1.1) is true for all positive integers n. ▲

Example 1.2. Prove that $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}(n)(n+1)(2n+1)$ (1.2) for all positive integers n. For $n = 1$,

$$\frac{1}{6}(1)(2)(3) = 1^2.$$

The statement is true for $n = 1$. Next we assume that for $n = k$, it is true, i.e.

$$1^2 + 2^2 + \cdots + k^2 = \frac{1}{6}(k)(k+1)(2k+1).$$

Then for $n = k + 1$,

$$\begin{aligned} 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 &= \frac{1}{6}(k)(k+1)(2k+1) + (k+1)^2 \qquad \text{(By assumption)} \\ &= \frac{1}{6}(k+1)[k(2k+1) + (6k+6)] \\ &= \frac{1}{6}(k+1)[2k^2 + 7k + 6] \\ &= \frac{1}{6}(k+1)(k+2)(2k+3). \end{aligned}$$

Then, by Mathematical Induction, the statement (1.2) is correct for all positive integers n. ▲

1.3 Variation in the First Step

For the first step, we need a starting point that the statement is true. But it is not necessarily be 1. In domino, for example, all the remaining dominoes still fall no matter which domino to be first to nudge.

Let $S(n)$ denote a statement involving a variable n. Suppose

1. $S(k_0)$ is true for some positive integer $k_0 > 1$;

2. If $S(k)$ is true for some positive integer $k \geq k_0$, then $S(k+1)$ is also true.

Then $S(n)$ is true for all positive integers $n \geq k_0$.

Example 1.3. Prove that $2^n > n^2$ for all natural numbers $n \geq 5$. When $n = 5$,

$$2^5 = 32 > 25 = 5^2,$$

so the inequality holds for $n = 5$. Assume the statement is true for some positive integer $k \geq 5$, i.e.

$$2^k > k^2.$$

Consider $n = k + 1$,

$$\begin{aligned} 2^{k+1} &= 2(2^k) \\ &> 2k^2 \qquad \text{(By assumption)} \\ 2k^2 - (k+1)^2 &= k^2 - 2k - 1 \\ &= (k-1)^2 - 2 \\ &> 0 \\ 2k^2 &> (k+1)^2 \\ \text{i.e.,} \qquad 2^{k+1} &> (k+1)^2. \end{aligned}$$

The statement is true when $n = k + 1$. Hence, by the principle of mathematical induction, the statement is true for all positive integers $n \geq 5$. ▲

1.4 Variation in the Second Step

For the second step, we may do the induction proof by more than 1, say 2, previous statements. Correspondingly, we need more, say 2, beginning points.

Let $S(n)$ denote a statement involving a variable n. Suppose

1. $S(1)$ and $S(2)$ are true;
2. If $S(k)$ and $S(k+1)$ are true for some positive integer k, then $S(k+2)$ is also true.

Then $S(n)$ is true for all positive integers n.

Example 1.4. Prove that $(3+\sqrt{5})^n + (3-\sqrt{5})^n$ is an even integer for all natural numbers n. Let $f(n) = \alpha^n + \beta^n$ where $\alpha = 3 + \sqrt{5}$ and $\beta = 3 - \sqrt{5}$. When $n = 1$,

$$f(1) = 6 \qquad \text{and} \qquad f(2) = 28$$

are even integers. Assume $f(k)$ and $f(k+1)$ are both even integers for some positive integer k. Consider $n = k + 2$, note that α and β are roots of the quadratic equation $x^2 - 6x + 4 = 0$. So, we have $\alpha^2 = 6\alpha - 4$ and $\beta^2 = 6\beta - 4$, and thus

$$\begin{aligned} f(k+2) &= \alpha^{k+2} + \beta^{k+2} \\ &= \alpha^k\alpha^2 + \beta^k\beta^2 \\ &= \alpha^k(6\alpha - 4) + \beta^k(6\beta - 4) \\ &= 6(\alpha^{k+1} + \beta^{k+1}) - 4(\alpha^k + \beta^k) \\ &= 6f(k+1) - 4f(k). \end{aligned}$$

It follows from the assumption that $f(k+2)$ must also be an even integer when $4, 6, f(k), f(k+1)$ are all even. The statement is true when $n = k + 2$. Hence, by the principle of mathematical induction, $f(n)$ is an even integer for all natural numbers n. ▲

1.5 Paradox*

Let us make some jokes in logic!

1.5.1 Everyone is pretty much bald

Let us spend a moment and get clear on what induction is and how it works in concrete terms. We want to prove that everyone is pretty much bald by induction:

The base case is easy: If you have 1 hair on your head, then certainly you are pretty much bald.

Now suppose inductively that if you have n hairs on your head, then you are pretty much bald.

We need to show that the same is true for someone with $n + 1$ hairs on their head. But certainly if someone has only 1 more hair on their head than someone else who is pretty much bald, then that first person is also pretty much bald. This completes the induction.

1.5.2 All horses are the same colour

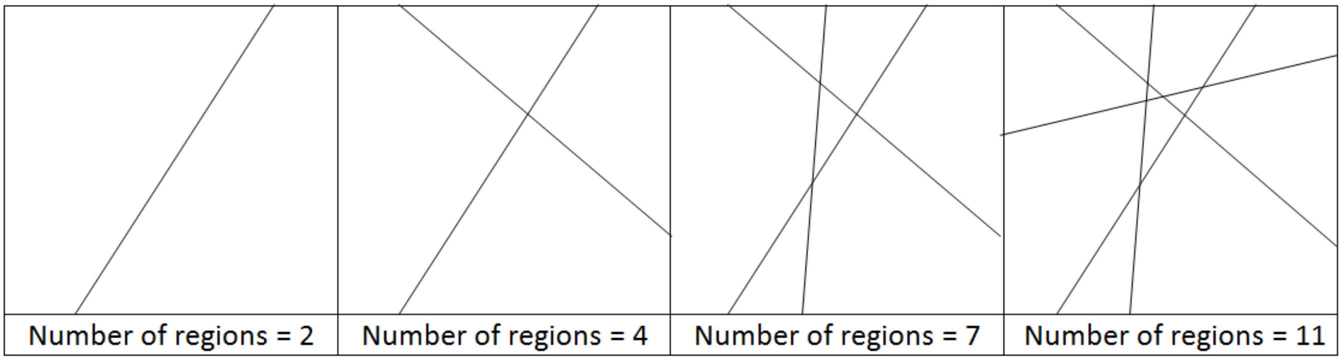
The first hurdle to jump is figuring out how to make this an inductive argument. But that is not too hard; I will translate the original claim into the following equivalent form: for every natural number n, every group consisting of n horses is monochromatic (i.e. they are all the same colour).

Now we can get our induction off the ground. The case $n = 1$ is obvious: every group consisting of 1 horse is monochromatic. Next comes the induction hypothesis: we are allowed to assume the result is true for n, and our job is to prove it true for $n + 1$. So consider a group of $n + 1$ horses. How can we show that they are all the same colour? Well, our inductive hypothesis tells us that every group of n horses is monochromatic. So all we have to do is remove one horse, call him $H1$, from our group of $n + 1$ horses and consider the remaining group; call it $G1$. There are n horses left in $G1$, so by the inductive hypothesis they are all the same colour. Now we have the problem that $H1$ is perhaps a different colour than the rest. But this problem too can be overcome: simply remove a different horse from the original group. This leaves behind a newgroup of n horses which includes $H1$; call it $G2$. Again we apply our inductive hypothesis, this time to deduce that $G2$ is monochromatic. But if $H1$ is the same colour as all the horses in $G2$, then $H1$ must be the same colour as a horse in $G1$, since every horse in $G2$ except $H1$ is also in $G1$. This then shows that $H1$ is the same colour as every horse in $G1$ (since $G1$ is monochromatic), and so the original group of $n + 1$ horses is monochromatic. This completes the induction and thereby proves that all horses are the same colour.

* Bjorndahl,Adam. Puzzles and Paradoxes in Mathematical Induction. http://www.math.cornell.edu/~mec/2008-2009/ABjorndahl/ppmi.pdf

1.6 Cool Induction Problems

Example 1.5. If n lines are drawn in a plane, and no two lines are parallel, what is the maximum number of regions that they separate the plane into? Let us see the first few cases, and have a guess.



The first four terms here are $2, 4, 7$ and 11. Guess if n lines are drawn in a plane, and no two lines are parallel, they separate the plane into $\dfrac{n^2+n+2}{2}$ regions. For $n = 1$, we can see on the diagram that there are 2 regions. So, it is true for $n = 1$. Assume if k lines are drawn in a plane, and no two lines are parallel, they separate the plane into $\dfrac{k^2+k+2}{2}$ regions. When $n = k+1$, on a plane with k non-parallel lines drawn, one more line is drawn. By assumption, there are $\dfrac{k^2+k+2}{2}$ regions separated by k non-parallel lines. As the $(k+1)$th line is not parallel to any line on the plane, it cuts k lines. To get the maximum number of regions, we assume the $(k+1)$th line does not cut any existing intersections. So the $(k+1)$th line divides $k+1$ regions each into 2, i.e. there are $k+1$ regions newly separated. Therefore, there are

$$\begin{aligned}\frac{k^2+k+2}{2}+(k+1) &= \frac{k^2+k+2+2k+2}{2}\\ &= \frac{(k^2+2k+1)+k+3)}{2}\\ &= \frac{(k+1)^2+(k+1)+2)}{2}\end{aligned}$$

regions. By Mathematical Induction, our guess is proved. More cool induction problems can be found in exercise. ▲

Exercises

In exercises $1-11$, use Mathematical Induction to show the statements are true.

1. Prove that $1+3+5+\cdots+2((n-1)-1)+(2n-1) = n^2$ for all positive integers n.
2. Prove that $1^2\times 2+2^2\times 3+\cdots+n^2\times(n+1) = \dfrac{(n)(n+1)(n+2)(3n+1)}{12}$ is true for all positive integers n.
3. Prove that $2002^{2n+1}+2003^{2n+1}$ is divisible by 4005 for all positive integers n.
4. Prove that

$$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n} \geq \frac{n}{2}$$

for all positive integers n.

5. Prove that $11^n - 6$ is divisible by 5 for all positive integers n.

6. Prove that $2^n > 2n$ for all positive integers $n > 2$.

7. Prove that $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{n \times (n+1)} = \frac{n}{n+1}$ for all positive integers n.

8. Prove that $n^3 - n$ is divisible by 6 for all positive integers n.

9. Prove that $2^{n+2} + 3^{2n+1}$ is divisible by 7 for all positive integers n.

10. Prove that $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\cdots\left(1 - \frac{1}{n}\right) = \frac{1}{n}$ for all positive integers n.

11. Prove that $\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{n-1}+\sqrt{n}} = \sqrt{n} - 1$ for all positive integers n.

12. **Challenging question 1**

    A circle and a chord of that circle are drawn in a plane. Then a second circle and a chord of that circle, are added. Repeating this process, once there are n circles with chords drawn, prove that the regions in the plane divided off by the circles and chords can be coloured with three colours in such a way that no two regions sharing some length of border are the same colour.

13. **Challenging question 2**

    Suppose $2n$ dots are placed around the outside of the circle. Among them, n are coloured red and the remaining n are coloured blue. Going around the circle clockwise, you keep a count of how many red and blue dots you have passed. If at all times the number of red dots you have passed is at least the number of blue dots, you consider it a successful trip around the circle. Prove that no matter how the dots are coloured red and blue, it is possible to have a successful trip around the circle if you start at the correct point.

14. **Challenging question 3**

    A sphere is covered with some number of "caps" which are hemispheres. Prove that it is possible to choose four hemispheres, and remove all others, while still keeping the sphere covered. (Hint: Sometimes it is easier to prove a more general statement than the one given.)

Chapter 2

Binomial Theorem

The binomial theorem (or binomial expansion) describes the algebraic expansion of the powers of a binomial. It is an easy technique to expand $(x+y)^n$ but applicable to many daily uses.
In Economics, economists use the binomial theorem to count probabilities that depend on numerous and very distributed variables to predict the way the economy will behave in the next few years. In Architecture, It allows engineers to calculate the magnitudes of the projects and thus deliver accurate estimates of not only the costs but also time required to construct them. For contractors, it is a very important tool to help ensuring the costing projects is competent enough to deliver profits. In computer science, binomial theorem has been very useful in distributing IP addresses, given that IP addresses is facing a shortage problem due to overpopulation. More other daily examples can be discovered online*.

2.1 Expanding a Polynomial

Remember that a binomial is a polynomial with two terms, e.g., $a+b, x^2+1, 2y-x$. Let us take a look at the first 3 powers of $a+b$:

$$\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= (a+b)(a^2+2ab+b^2) = a^3+3a^2b+3ab^2+b^3
\end{aligned}$$

Recall that $x^0 = 1$ for all real numbers x, including $x = 0$, i.e. $0^0 = 1$. Well, expanding binomials is tedious, especially when it comes to a large power. Indeed, there is a way to determine the expanded terms of a binomial, known as the binomial theorem.

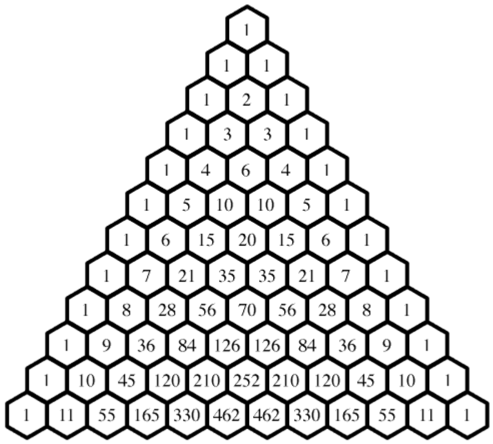
2.2 Pascal's Triangle

In elementary algebra, the binomial theorem (or binomial expansion) describes the algebraic expansion of powers of a binomial. According to the theorem, it is possible to expand the power $(x+y)^n$ into a sum

*https://www.quora.com/What-are-some-real-world-examples-of-the-use-of-the-binomial-theorem involving terms of the form $ax^b y^c$, where the exponents b and c are nonnegative integers with $b+c=n$, and the coefficient a of each term is a specific positive integer depending on n and b. For example,

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

These coefficients for varying n and b can be arranged to form Pascals Triangle[†].



The binomial theorem says it is possible to expand any integer power of $x+y$ into a sum of the form

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n,$$

where each $\binom{n}{r}$ is a specific positive integer known as a binomial coefficient (refer to the Pascal's Triangle above).

2.3 Proof of Binomial Theorem

Mathematical Induction yields a proof of the binomial theorem, but we need some basic concepts first.

2.3.1 Factorial

In mathematics, the factorial of a non-negative integer n, denoted by $n!$, is the product of all positive integers less than or equal to n,

$$n! = n \times (n-1) \times (n-2) \cdots \times 1,$$

and 0! is defined as 1.

Example 2.1.

$$3! = 3 \times 2 \times 1 = 6$$

▲

Example 2.2.

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

▲

[†]`https://en.wikipedia.org/wiki/Binomial_theorem`

2.3.2 Binomial Coefficient

For any binomial coefficients $\binom{n}{r}$, also known as C_r^n, of nonnegative integers n and r,

$$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

This family of numbers also arises in many areas of mathematics other than algebra, especially in combinatorics. Both C_r^n and $\binom{n}{r}$ are often read aloud as "n choose r", because there $\binom{n}{r}$ ways to choose r elements, disregarding their order, from a set of n elements. For any integer $n > 0$, in case of $r < 0$ or $r > n$, $\binom{n}{r}$ is defined as 0.

Example 2.3. Calculate the value of $\binom{9}{3}$.

$$C_3^9 = \binom{9}{3} = \frac{9!}{3!(9-3)!} = \frac{9 \times 8 \times 7 \times 6!}{3 \times 2 \times 6!} = 84$$

▲

2.3.3 Exponential

The exponent of a number corresponds to the repeated multiplication of that number[‡]. If we wish to multiply an integer x (we call it the base) by n (we call it the exponent) times, we express in $x^n = x \cdot x \cdot x \cdots x$.

Example 2.4. Find 1.02^3 by binomial theorem.

$$\begin{aligned}(1+0.02)^3 &= 1^3 + 3(1)^2(0.02) + 3(1)(0.02)^2 + (0.02)^3 \\ &= 1.061208\end{aligned}$$

▲

Example 2.5. Find $(2+\sqrt{2})^2$.

$$\begin{aligned}(2+\sqrt{2})^2 &= 2^2 + 2(2)(\sqrt{2}) + (\sqrt{2})^2 \\ &= 6 + 4\sqrt{2}\end{aligned}$$

▲

2.3.4 Summation

In mathematics, summation is the addition of a sequence or list of numbers. The result is their sum or total. The numbers in the list to be summed may be integers, rational numbers, real numbers or complex numbers. It is a shorthand to sum up a long addition to numbers, i.e.

$$\sum_{j=1}^{n} f(j) = f(1) + f(2) + \cdots + f(n).$$

[‡] `https://en.wikipedia.org/wiki/Exponentiation`

Example 2.6. Use the summation notation to write $1+2+3+\cdots+100$.

$$1+2+3+\cdots+100=\sum_{j=1}^{100} j$$

The j here is a dummy variable that can be replaced by any other variables, say, i or k. $\sum_{i=1}^{100} i=\sum_{j=1}^{100} j$. ▲

Example 2.7. Find $\displaystyle\sum_{j=0}^{4} 2^j$.

$$\begin{aligned}\sum_{j=0}^{4} 2^j &= 2^0+2^1+2^2+2^3+2^4\\ &= 31\end{aligned}$$

▲

Example 2.8. Prove that $T(n): \displaystyle\sum_{j=1}^{n} j=\frac{n(n+1)}{2}$ for all positive integers n by mathematical induction.

When $n=1$,

$$\begin{aligned}L.H.S. &= \sum_{j=1}^{1} j=1,\\ R.H.S. &= \frac{1(1+1)}{2}=1.\end{aligned}$$

So, $T(1)$ holds. Then, let us assume $T(k)$ is true for some positive integers k, i.e.

$$\sum_{j=1}^{k} j=\frac{k(k+1)}{2}.$$

For $n=k+1$,

$$\begin{aligned}L.H.S. &= \sum_{j=1}^{k+1} j\\ &= \sum_{j=1}^{k} j+(k+1)\\ &= \frac{k(k+1)}{2}+(k+1)\\ &= (k+1)\left(\frac{k+2}{2}\right)\\ &= \frac{(k+1)(k+2)}{2}\\ &= R.H.S.\end{aligned}$$

So $T(k+1)$ holds as well and the induction is completed. ▲

Using the summation notation, the binomial theorem can be written as

$$(x+y)^n=\sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j.$$

We can prove the binomial theorem by Mathematical Induction now.

Proof of the Binomial Theorem. [§]
For $n = 1$,

$$L.H.S. = (x+y)^1 = x+y = \binom{1}{0}x^{1-0}y^0 + \binom{1}{0}x^{1-1}y^1 = \sum_{j=0}^{1}\binom{1}{j}x^{1-j}y^j = R.H.S.$$

The binomial theorem is true for $n = 1$. Assume the statement is true for some integer k, i.e.,

$$(x+y)^k = \sum_{j=0}^{k}\binom{k}{j}x^{k-j}y^j.$$

Consider $n = k+1$,

$$\begin{aligned}
(x+y)^{k+1} &= (x+y)(x+y)^k \\
&= (x+y)\left[\sum_{j=0}^{k}\binom{k}{j}x^{(k-j)}y^j\right] \\
&= \sum_{j=0}^{k}\binom{k}{j}x^{k+1-j}y^j + \sum_{m=0}^{k}\binom{k}{m}x^{(k-m)}y^{m+1} \\
&= \sum_{j=0}^{k}\binom{k}{j}x^{k+1-j}y^j + \sum_{m=0}^{k}\binom{k}{(m+1)-1}x^{(k+1)-(m+1)}y^{m+1} \\
&= \sum_{j=0}^{k}\binom{k}{j}x^{k+1-j}y^j + \sum_{j=1}^{k+1}\binom{k}{j-1}x^{k+1-j}y^j \\
&= \sum_{j=0}^{k+1}\left[\binom{k}{j}x^{k+1-j}y^j\right] - \binom{k}{k+1}x^0y^{k+1} + \sum_{j=0}^{k+1}\left[\binom{k}{j-1}x^{k+1-j}y^j\right] - \binom{k}{-1}x^{k+1}y^0 \\
&= \sum_{j=0}^{k+1}\left[\binom{k}{j} + \binom{k}{j-1}\right]x^{k+1-j}y^j \\
&= \sum_{j=0}^{k+1}\binom{k+1}{j}x^{k+1-j}y^j
\end{aligned}$$

And the binomial theorem is proved! ■

Exercises

1. Write each of the following sums using the $\sum$ notation.
   (a) $1 + 4 + 9 + 16 + 25 + 36$
   (b) $3 - 5 + 7 - 9 + 11 - 13 + 15$
   (c) $2x^3 + 4x^5 + 6x^7 + \cdots + 30x^{31}$
   (d) $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!}$
2. Use mathematical induction to prove the following formula holds for all positive integers m

$$\sum_{j=1}^{m} j^3 = \frac{m^2(m+1)^2}{4}.$$

[§] `https://www.math.hmc.edu/calculus/tutorials/binomial_thm/induction.html`

3. Evaluate each sum.
   (a) $\sum_{j=1}^{n} 2j(j-1)$
   (b) $\sum_{j=1}^{n} 3j(2-j^2)$
   (c) $\sum_{j=1}^{n} j(7j+3)^2$

4. Simplify the following expressions.
   (a) $\frac{(n+2)!}{n!}$
   (b) $\frac{(2n+2)!}{2n!}$
   (c) $\frac{(n-1)!}{(n+1)!}$

5. (a) Evaluate $\frac{10!}{5!}$.
   (b) Simplify $\frac{(n+1)!}{n!}$.

6. Given that $C_r^{n+1} = C_r^n + C_{r-1}^n$. Prove that $\sum_{j=0}^{n} (C_j^n)^2 = C_n^{2n}$.

7. Prove that $C_0^n + C_1^n + \cdots + C_{n-1}^n + C_n^n = 2^n$.

8. Prove that $0 \cdot C_0^n + 1 \cdot C_1^n + 2 \cdot C_2^n + \cdots + n \cdot C_n^n = n \cdot 2^{n-1}$.

9. Expand the following using the binomial theorem.
   (a) $(2x-3)^4$
   (b) $\left(x - \frac{2}{x}\right)^6$

10. (a) Prove that $C_{r+1}^n = C_r^{n-1} + C_{r+1}^{n-1}$.
    (b) Let n and r be integers with $0 \leq r \leq n$. Prove that $C_{r+1}^n = C_r^n + C_r^{n-1} + \cdots + C_r^r$.

11. Expand $(x-2)^{10}$ in ascending powers of x, up to the term containing x^3.

12. Find the coefficient of x^5 in the expansion of $(2x-3)^7$.

13. Find the coefficient of x^3 in $(1+x+2x^2)(1-2x)^5$.

14. Is it possible for two consecutive terms in the expansion of $(2x+3)^9$ to have equal coefficients? If so, find them.

15. Let n and r be integers with $0 \leq r \leq n$. Find $\sum_{j=1}^{n} jC_j^n$. Hence find $\sum_{j=1}^{n} j^2 C_j^n$.

16. Prove that $\sum_{j=1}^{n} r^j = \frac{1-r^{n+1}}{1-r}$, for r $\neq 1$ for all positive integers n.

17. Prove that the constant term in the expansion of $\left(x^4 - \frac{2}{x}\right)^n$ is non-zero if and only if[¶] n is a multiple of 5, and the constant term is never negative.

18. Find all positive integers a, b so that in the expression $(1+x)^a + (1+x)^b$, the coefficients of x and x^2 are equal.

19. Prove, by mathematical induction, that $1 \times 1! + 2 \times 2! + \cdots + n \times n! = (n+1)! - 1$ is true for all positive integers n.

20. (a) Expand $(k + 4x)^5$ in ascending powers of x as far as the term in x^3 where k is a constant.
(b) If the coefficient of x^3 in (a) is 640, find the values of k.

21. Find the maximum coefficient in the expansion of $(3x + 5)^{10}$ without actual expansion.

22. (a) Find the general term in the expansion of $\left(\frac{1}{x^2} - 2x^2\right)^{10}$.
(b) Find the coefficient of $\frac{1}{x^4}$.
(c) Find the constant term.
(d) Does the expansion of $\left(\frac{1}{x^2} - 2x^2\right)^{10}$ contain the term in x^3?

23. (a) Find the general term in the expansion of $\left(x^2 - \frac{1}{x}\right)^7$.
(b) Find the coefficient of $\frac{1}{x^4}$.
(c) Does the expansion of $\left(x^2 - \frac{1}{x}\right)^7$ contain the term in x^{-1}?

24. It is given that $(1-2x)^2(1+x)^n = a + bx + cx^2 + \cdots$ where a, b and c are constants.
(a) Find a.
(b) Express b and c in terms of n.
(c) If the coefficient of x^2 is -5, find the values of n.

[¶] if and only if (iff) is a biconditional way of writing two conditionals at once: both a conditional and its converse.

Chapter 3

Arithmetic and Geometric Series

Sequence is a list of numbers in a particular pattern, that one can guess the next number by knowing what the former ones are. For an instance, if you are given a sequence which starts with the numbers $1, 2, 3, \cdots$, it is clear that the sequence represents the natural numbers and the fourth number is intuitively 4. Another example is $0, 3, 6, 9, 12, 15, \cdots$. One can interpret that it is a list of numbers adding 3 each time on top of the previous one. So we can guess that the next number should be 18[*].

Definition 3.1
A sequence is an ordered list of numbers. According to the ordered list, the notation of the nth term is $T(n)$. And a sequence is denoted by

$$T(1), T(2), T(3), \cdots, T(k), \cdots$$

Another common notation for a sequence is $\{a_n\}$ for all positive integers n, where $a_1, a_2, \cdots, a_k, \cdots$ are the terms of the sequence.

Example 3.1. Positive integers "$1, 2, 3, 4, \cdots$" form an infinite sequence, with terms $T(1) = 1$, $T(2) = 2$, $T(3) = 3$, $T(4) = 4$ and so on. $T(n) = n$ for all positive integers n. ▲

When we have a sequence (a list of numbers), we can add them up: either partially till the nth term, or add up infinitely terms. There are some sequences that follow a pattern and we can express the items by a general formula. We will talk about series which is the total of a sequence.

3.1 Arithmetic Sequence and Arithmetic Series

3.1.1 Arithmetic Sequence

An arithmetic sequence is a sequence in which the difference between two consecutive terms (except the first term) and its preceding term is a constant[†]. The general term $T(k)$ of an arithmetic sequence with first term a and common difference d is given by

$$T(k) = a + (k - 1)d.$$

[*] `http://www.coolmath.com/algebra/19-sequences-series`
[†] Wong, Ka-ming, Chun-wah Hung, Kwok-cheung Mak, and Kwok-kwan Chan. Mathematics in Focus. Hong Kong: Educational House, 2013.

Take the aforementioned example,

$$0, 3, 6, 9, 12, 15, 18, \cdots$$

Clearly the initial term $a = 0$, and the common difference $d = 3$. Therefore $T(k) = 0 + (k-1)3 = 3k - 3$.

3.1.2 Arithmetic Partial Sum

As adding up infinitely terms of arithmetics will always go to infinity, we are not interested in it. Instead, **partial sum**, the sum of part of the sequence from the first term to the nth term is more worthwhile to analyze. The arithmetic partial sum with the first term a can be found by

$$S_n = \frac{n}{2}\left[a + T(n)\right].$$

If we put $T(n) = a + (n-1)d$ into the formula, we have

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right].$$

Proof. We can express the arithmetic series in two different ways, i.e.,

$$\begin{aligned} S_n &= a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + [a_1 + (n-2)d] + [a_1 + (n-1)d] \\ S_n &= [a_n - (n-1)d] + [a_n - (n-2)d] + \cdots + (a_n - 2d) + (a_n - d) + a_n. \end{aligned}$$

Adding both sides of the two equations, all terms involving d cancel:

$$2S_n = n(a_1 + a_n).$$

Dividing both sides by 2 produces a common form of the equation:

$$S_n = \frac{n}{2}(a_1 + a_n).$$

An alternate form results from re-inserting the substitution: $a_n = a_1 + (n-1)d$:

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right].$$

∎

Another Proof. Mathematical Induction yields another proof of the arithmetic partial sum. For $n = 1$,

$$L.H.S. = S_1 = a_1 = \frac{1}{2}(2a_1) = R.H.S.$$

It is true for $n = 1$. Then, assume the statement is true for some integer k, i.e.,

$$S_k = \frac{k}{2}\left[2a_1 + (k-1)d\right].$$

is true. Consider $n = k + 1$,

$$\begin{aligned} S_{k+1} &= S_k + T_{k+1} \\ &= S_k + a_1 + kd \\ &= S_k + a_1 + \frac{k}{2}d + \frac{k}{2}d \\ &= \frac{k}{2}\left[2a_1 + (k-1)d\right] + \frac{k}{2}d + \frac{1}{2}(2a_1 + kd) \\ &= \frac{k}{2}\left[2a_1 + (k-1)d + d\right] + \frac{1}{2}(2a_1 + kd) \\ &= \left(\frac{k+1}{2}\right)(2a_1 + kd). \end{aligned}$$

By Mathematical Induction, the statement is true for all positive integers n. ■

Example 3.2. In an arithmetic sequence, n terms are given: $16, 13, 10, \cdots, -71$.
(a) Find n.
(b) Find the sum of all the terms of the sequence.
(c) Find the sum from the 21st term to the 30th term.

(a) The first term $a = 16$, the common difference $d = T(2) - T(1) = 13 - 16 = -3$.
Suppose there are k terms in the sequence.

$$\begin{aligned} T(k) &= -71 \\ &= 16 + (k-1)(-3) \\ k &= 30. \end{aligned}$$

Hence, there are 30 terms in the sequence.

(b) Since there are 30 terms in the sequence.

$$\begin{aligned} S(30) &= \frac{30}{2}\left[16 + (-71)\right] \\ &= -825. \end{aligned}$$

(c) $S(30)$ is found in part (b),

$$\begin{aligned} S(20) &= \frac{20}{2}\left[16 + 16 + (19)(-3)\right] \\ &= -250 \\ S(30) - S(20) &= -825 - (-250) \\ &= -575. \end{aligned}$$

Hence, the required sum is -575. ▲

Example 3.3. Prove that the series of n terms of an arithmetic sequence with $a = 1$ and $d = 2$ is n^2.

The required statement is $1 + 3 + 5 + \cdots + (2n-1) = n^2$, for all natural numbers n.

When $n = 1$, the statement is trivial since $1 = 1^2$. Assume the statement is true for some positive integer k, i.e.,

$$1 + 3 + 5 + \cdots + (2k-1) = k^2.$$

Consider $n = k + 1$,

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k-1) + (2(k+1)-1) &= k^2 + (2k+1) \qquad \text{(By assumption)} \\ &= (k+1)^2. \end{aligned}$$

The statement is true when $n = k + 1$. Hence, by the principle of mathematical induction, the statement is true for all natural numbers n. ▲

3.2 Geometric Sequence and Geometric Series

3.2.1 Geometric Sequence

A geometric sequence is a sequence in which the ratio of any two consecutive terms (except the first term) is a constant**.

The general term $T(k)$ of a geometric sequence with first term a and common ratio r is given by

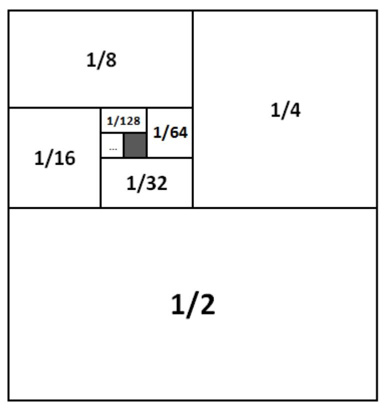
$$T(k) = ar^{k-1}.$$

The relationships between two consecutive terms is given by

$$T(k+1) = r \cdot T(k).$$

Example 3.4.

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$$



Clearly the initial term $a = 1$, and the common ratio $r = \frac{1}{2}$. Therefore $T(k) = \left(\frac{1}{2}\right)^{k-1}$. ▲

3.2.2 Geometric Series

For a geometric sequence with first term $a_1 = a$ and common ratio r, the partial sum of the first n terms is given by

$$S(n) = \sum_{j=1}^{n} ar^{j-1} = \frac{a(1-r^n)}{1-r}, \qquad \text{where } r \neq 1.$$

Different from arithmetic series, except the sum of the finite number of terms, we are also interested in the sum of infinite number of terms, with common ratio r be between -1 and 1.
Let a be the first term, and r be the common ratio, then the infinite geometric series can be found by

$$S(\infty) = \frac{a}{1-r}, \qquad \text{where } -1 < r < 1.$$

**Wong, Ka-ming, Chun-wah Hung, Kwok-cheung Mak, and Kwok-kwan Chan. Mathematics in Focus. Hong Kong: Educational House, 2013.

Proof. We can derive the sum of the first n terms of a geometric series, i.e. $S(n) = \sum_{j=1}^{n} ar^{j-1} = \frac{a(1-r^n)}{1-r}$, as follows:

$$\begin{aligned} s &= a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \\ rs &= ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n \\ s - rs &= a - ar^n \\ s(1-r) &= a(1-r^n) \\ s &= a\frac{1-r^n}{1-r} \qquad \text{(if } r \neq 1\text{).} \end{aligned}$$

As n goes to infinity, the absolute value of r must be less than 1 for the series to converge. The sum then becomes

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots = \sum_{j=0}^{\infty} ar^j = \frac{a}{1-r}, \qquad \text{for } |r| < 1.$$

■

Another Proof. Mathematical Induction yields another proof of the geometric partial sum. For $n = 1$,

$$\begin{aligned} S(1) &= ar^{1-1} \\ &= a \\ &= \frac{a(1-r^1)}{1-r}. \end{aligned}$$

So, $S(1)$ holds. Assume the statement is true for some positive integer k, i.e.

$$S(k) = \sum_{j=1}^{k} ar^{j-1} = \frac{a(1-r^k)}{1-r}.$$

Then for $n = k+1$,

$$\begin{aligned} S(k+1) &= \sum_{j=1}^{k+1} ar^{j-1} \\ &= \frac{a(1-r^k)}{1-r} + ar^k \\ &= \frac{a(1-r^k)}{1-r} + \frac{(1-r)ar^k}{1-r} \\ &= \frac{a - ar^k + ar^k - ar^{k+1}}{1-r} \\ &= \frac{a(1-r^{k+1})}{1-r}. \end{aligned}$$

By Mathematical Induction, the statement is true for all positive integer n. ■

Example 3.5. To convert $0.\dot{2}$ into a fraction, notice that

$$0.\dot{2} = 0.2 + 0.02 + 0.002 + \cdots$$

which is an infinite geometric series with beginning term $a = 0.2$, and $r = \frac{0.02}{0.2} = 0.1$. Then,

$$0.\dot{2} = S(\infty) = \frac{0.2}{1-0.1} = \frac{2}{9}.$$

▲

3.3 Fibonacci Sequence and Fibonacci Series

3.3.1 Fibonacci Sequence‡

The Fibonacci numbers are the sequence of numbers F_n defined by the linear recurrence equation

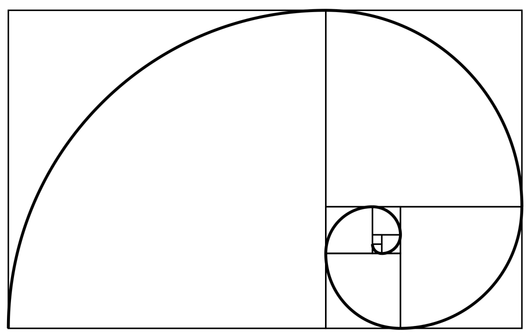
$$F_n = F_{n-1} + F_{n-2} \quad \text{with} \quad F_1 = F_2 = 1. \quad n \geq 3$$

As a result of the definition, the recurrence equation generates the following Fibonacci numbers:

$$1, 1, 2, 3, 5, 8, 13, 21, \cdots$$

3.3.2 Golden Spiral(Ratio)

If you sum the square of any series of Fibonacci numbers, a **Fibonacci spiral** will be generated as follows.



The Fibonacci spiral ideally gets closer and closer to a golden spiral(ratio), as it increases in size because of the ratio of each number in the Fibonacci series to the one before it converges to $\Phi, 1.618\cdots$, as the series progresses. (e.g., $1, 1, 2, 3, 5, 8$ and 13 produce ratios of $1, 2, 1.5, 1.67, 1.6$ and 1.625 respectively)

3.3.3 Fibonacci Series

The sum of the Fibonacci sequence is easy to evaluate. Given that

$$\begin{aligned} F_n &= F_{n+2} - F_{n+1} \\ F_{n-1} &= F_{n+1} - Fn \\ &\vdots \\ F_1 &= F_3 - F_2. \end{aligned}$$

Hence, taking the sum of the equations above. We have $\sum_{j=1}^{n} F_j = F_{n+2} - F_2 = F_{n+2} - 1$. So the series is closely related to the sequence and does not require a separated study.

3.3.4 Application of Fibonacci Number

In computer science, **Fibonacci search** is a searching method based on Fibonacci sequence. It is a method of searching a sorted array using the divide and conquer algorithm that narrows down possible locations

‡ `http://mathworld.wolfram.com/FibonacciNumber.html`

with the aid of Fibonacci numbers.
In design industry, golden ratio is widely applied in designing things since people believe creations according to the golden ratio becomes more beautiful.
In fact, the golden spiral is everywhere around us. Spirals are common in nature and have inspired mathematicians for centuries. For example, it appears in the number of petals in a flower. Famous examples include the lily, which has three petals, buttercups, which have five, the chicory's 21, the daisy's 34, and so on. Φ appears in petals on account of the ideal packing arrangement, that each petal is placed at a perfect position allowing for the best possible exposure to sunlight and other factors[§].

Exercises

1. Find the sum of the first 50 terms of the sequence $1, 3, 5, 7, 9, \cdots$.

2. Find the series $1 + 3.5 + 6 + 8.5 + \cdots + 101$.

3. Consider the Fibonacci sequence. By induction, prove that F_{3m} (that is, every third Fibonacci number - $F_1, F_3, F_6, \cdots$) is even for every integer $n \geq 1$.

4. Prove that for all natural numbers n, $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$.

5. Let $\{a_n\}$ be a sequence of natural numbers such that $a_1 = 5, a_2 = 13$ and $a_{n+2} = 5a_{n+1} - 6a_n$ for all natural numbers n. Prove that $a_n = 2^n + 3^n$ for all natural numbers n.

6. An arithmetic sequence has 3 as its first term. Also, the sum of the first 8 terms is twice the sum of the first 5 terms. Find the common difference.

7. The sum of the first 20 terms of an arithmetic series is identical to the sum of the first 22 terms. If the common difference is -2, find the first term.

8. Let p, q and r be the lengths of three sides of $\triangle PQR$. If p, q, r form an arithmetic sequence and the quadratic equation $px^2 + 2qx + r = 0$ has equal roots, prove that $\triangle PQR$ is an equilateral triangle.

9. Suppose 40 sandwiches are packed into four batches A,B,C and D. There are 4 sandwiches in batch D. The numbers of sandwiches in batches A,B,C and D form an arithmetic sequence. If batch D contains the least number of sandwiches, how many sandwiches are there in each of batches A,B and C?

10. The common difference and the sum of the first 15 terms of an arithmetic sequence are $\frac{3}{2}$ and 240 respectively.
    (a) Find the first term of the sequence.
    (b) If the sum of the first k terms of the sequence is 361, find the value of k.

11. In the first minute, 5 users logged on to play an online game. The number of users logging on in each succeeding minute is 2 more than that in the previous minute. Find the total number of users logging on to play the game after half an hour.

[§]https://www.quora.com/What-are-the-real-life-applications-of-Fibonacci-series

12. The sum of the first 3 terms of a geometric series is $\frac{37}{8}$. The sum of the first six terms is $\frac{3367}{512}$. Find the first term and common ratio.

13. How many terms in the geometric sequence $4, 3.6, 3.24, \cdots$ are needed so that the sum exceeds 35?

14. The sum to infinity of a geometric sequence is twice the sum of the first two terms. Find possible values of the common ratio.

15. (a) Find the general term $T(n)$ of the geometric sequence $14, 28, 56, \cdots$
    (b) Using the result of (a), write down the general term $Q(n)$ of the sequence $\log 14, \log 28, \log 56 \cdots$ Hence, find $Q(n+1) - Q(n)$ and determine whether this sequence is an arithmetic sequence.

16. In a geometric sequence, the 5th term is greater than the 3rd term by 72. The 3rd term is 9 times the 1st term.
    (a) Find the possible general term(s) $T(n)$ of the sequence.
    (b) Using the result of (a), how many terms are there between 1 and 2500 inclusive?

17. If the sum of the first k terms of the geometric sequence $27, 36, 48, \cdots$ is smaller than 1000, find the largest value of k.

18. The *Lucas sequence* $1, 3, 4, 7, 11, 18, 29, \cdots$ is defined by

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 3 \\ a_n &= a_{n-1} + a_{n-2}, \qquad \text{for } n \geq 3. \end{aligned}$$

    Prove, by mathematical induction, that $a_n < (1.75)^n$ for all positive integers n.

19. The sequence of *Catalan numbers* $1, 1, 2, 5, 14, 42, \cdots$ is given by the formula

$$C_n = \frac{1}{n+1} C_n^{2n}, \qquad n = 0, 1, 2, \cdots.$$

    Prove that $C_n = \frac{2(2n-1)}{n+1} C_{n-1}$ for all positive integers n.

20. Suppose $\$P$ is deposited in a bank at an interest rate of $r\%$ per annum, compounded yearly. At the end of each year, $\frac{2}{5}$ of the amount is withdrawn and the remainder is left at the same interest rate. Let $\$R(n)$ be the sum of money withdrawn at the end of the nth year.

    (a) i. Express $R(1)$ and $R(2)$ in terms of P and r.
        ii. Show that $R(3) = \frac{18}{125} P(1 + r\%)^3$.
    (b) Given that $R(1), R(2), R(3), \cdots$ is a geometric sequence, express the common ratio in terms of r.
    (c) i. Let $R(3) = \frac{16}{81} P$. Find the value of r.
        ii. If $P = 50000$, find $R(1) + R(2) + R(3) + \cdots + R(10)$, correct to 3 significant figures.

21. Peter borrows a loan of \$200000 from a bank at an interest rate of 6% per annum, compounded monthly. For each successive month after the day when the loan is taken, loan interest is calculated and then a monthly instalment of $\$x$ is immediately paid to the bank until the loan is fully repaid (the last instalment may be less than $\$x$), where $x < 200000$.

    (a) i. Find the loan interest for the 1st month.

    ii. Express, in terms of x, the amount that Peter still owes the bank after paying the 1st instalment.

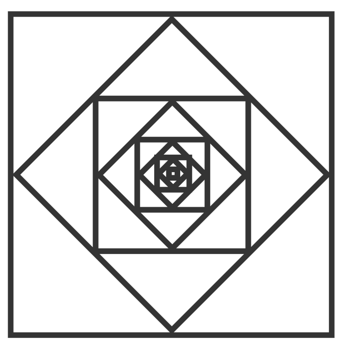
    iii. Prove that if Peter has not yet fully repaid the loan after paying the nth instalment, he still owes the bank $\$\{200000(1.005)^n - 200x[(1.005)^n - 1]\}$.

    (b) i. Suppose that Peter's monthly instalment is \$1800 (the last instalment may be less than \$1800). Find the number of months for Peter to fully repay the loan.

    ii. Peter wants to fully repay the loan with a smaller monthly instalment. He requests to pay a monthly instalment of \$900. However, the bank refuses his request. Why?

22. **Challenging question 1**

    Suppose that you draw a $2cm$ by $2cm$ square, then you join the midpoints of its sides to draw another square, then you join the midpoints of that square's sides to draw another square, and so on, as shown below.



Would you need infinitely many pens to continue this process forever? i.e., does drawing the squares infinitely require an infinite amount of ink to do so?