Chapter 9

Limit

Limit is the foundation of calculus that it is so useful to understand more complicating chapters of calculus. Besides, Mathematics has "black hole" scenarios (dividing by zero, going to infinity etc.), and limits give us an estimate when we can't compute a result directly. Limits allow us to ask "What if?". If we can directly observe a function at a value, we don't need a prediction. The limit wonders, "If you can see everything *except* a single value, what do you think is there?".*

9.1 Introduction

Sometimes we cannot work something out directly, but we can see what it should be as we get closer and closer. To makes the concept of limit clear, an example will do.

Example 9.1. For $x = 1$, what is the value of the function

$$y = \frac{x^2 - 1}{x - 1}$$

For $x = 1$,

$$\begin{aligned} y &= \frac{1^2 - 1}{1 - 1} \\ &= \frac{0}{0} \end{aligned}$$

Now, $\frac{0}{0}$ is difficult for us to understand that we do not really know the value of $\frac{0}{0}$ (it is "indeterminate"), so we need another way of answering this.

So instead of trying to work it out for $x = 1$, try approaching it closer and closer:

*https://betterexplained.com/articles/an-intuitive-introduction-to-limits/

x	$\frac{x^2-1}{x-1}$
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999
0.9999	1.9999

Now we see that as x gets close to 1, then $\frac{x^2-1}{x-1}$ gets close to 2. Before we proceed to the next step, we shall try if we can still obtain the same result, starting from the larger side of x. Let us see.

x	$\frac{x^2-1}{x-1}$
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001
1.0001	2.0001

Turns out the results from the left (i.e. smaller x) and the right (i.e. larger x) yield the same outcome that the function converges to 2 as well. We are more certain that it converges to 2 when x approaches 1.

We want to say that $y|_{x=2} = \frac{(1)^2-1}{(1)-1} = 2$ but we can't, so instead mathematicians say exactly what is going on by using the magic word "limit". † ▲

9.2 Definition

If $f(x)$ is defined for all x near c, except possibly at c itself, and if we can ensure that $f(x)$ is as close as we want to L by taking x close enough to (but different from) c, we say that the function f approaches the **limit** L as x approaches c, and we write it as

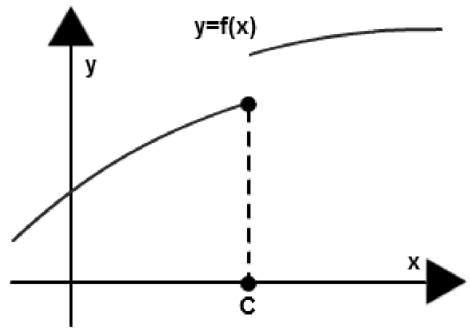
$$\lim_{x\to 1} f(x) = L.$$

Therefore, for the above example, we shall express the result as

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2.$$

Then, we shall ask: what if the results from the left and right are not the same? What does it interpret? Does the limit still exist?

† https://www.mathsisfun.com/calculus/limits.html



From the graph above, we can see that the curve is not continuous (broken) at c. If we adopt the approach used in the previous example, we will without doubt obtain different outcomes from the left and from the right. We express them as

$$\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x).$$

We say the limit **does not exist** at $x = c$ if the left limit and right limit are not equal.

9.3 Properties of Limit

There are many properties to help calculate the values of limits. If $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$, and k is a constant, then

1. $\lim_{x\to c} kf(x) = kL$
2. $\lim_{x\to c}[f(x) \pm g(x)] = L \pm M$
3. $\lim_{x\to c}[f(x) \cdot g(x)] = LM$
4. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, for$M \neq 0$
5. If $f(x) \leq g(x)$ on an interval containing c in its interior, then the order of the limits is preserved, i.e., $L \leq M$.

Example 9.2. Find $\lim_{x\to 1} \frac{x^2-1}{x-1}$

$$\begin{aligned}
\lim_{x\to 1} \frac{x^2-1}{x-1} &= \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} \\
&= \lim_{x\to 1}(x+1) \\
&= \lim_{x\to 1} x + \lim_{x\to 1} 1 \\
&= 1+1 \\
&= 2
\end{aligned}$$

▲

9.4 Technique

Limit problems could be quite tacky sometimes that require more advanced techniques to deal with, some useful skills are listed as follows.

9.4.1 The Squeezing Theorem

The Squeezing Theorem, also known as the Sandwich Theorem, helps predict the value of a limit, by knowing the limits of its boundaries. For example, we know that a boy is walking on the street, with a bakery shop at the start of the street, a book store in the middle and an hospital at the end. Assuming that the boy goes into the shops agreed to their order, if we know that the boy goes into the bakery shop at $16:02$ and into the hospital at $16:02$, then it is certain that he arrives the book store at $16:02$ as well.
In mathematical terms, suppose that $f(x) \leq g(x) \leq h(x)$ holds for all x in some open interval containing a, except possibly at $x = a$ itself. Suppose also that

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

Then $\lim_{x \to a} g(x) = L$ too. Similar statements hold for left and right limits.

Example 9.3. Find the limit $\lim_{x \to 0} x^2 \sin(\frac{1}{x})$.
Consider that

$$-1 \leq \sin(\frac{1}{x}) \leq 1$$
$$-x^2 \leq x^2 \sin(\frac{1}{x}) \leq x^2$$

As

$$\lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0,$$

By squeezing theorem,

$$\lim_{x \to 0} x^2 \sin(\frac{1}{x}) = 0.$$

▲

9.4.2 Intermediate Value Theorem

A continuous function attains all values between any two of its values. Take the following example to explain what and how Intermediate Value Theorem works. A girl who weights 5 pounds at birth and 100 pounds at age 12 must have weighted exactly 50 pounds at some time during her 12 years of life, since her weight is a continuous function of time.
In mathematical terms, suppose that $f(x)$ is continuous on [a,b] and W is any number between $f(a)$ and $f(b)$. Then, there is at least one number $c \in [a, b]$ for which

$$f(c) = W.$$

Example 9.4. Show that the equation $x^2 - x - 1 = \frac{1}{x+1}$ has a solution between $1 \leq x \leq 2$.

Let

$$f(x) = x^2 - x - 1 - \frac{1}{x+1} = 0,$$

which means we have to prove that there exists c such that $f(c) = 0$.

Then

$$f(1) = -\frac{3}{2}, f(2) = \frac{2}{3}$$

By Intermediate Value Theorem, the curve must cross the x-axis somewhere between $x = 1$ and $x = 2$.

i.e. there is a number c such that $1 \leq c \leq 2$ and $f(c) = 0$.

▲

9.5 Exercises

Find the left and right limits of questions $1 - 4$.

1. $f(x) = \dfrac{x^2 - 4}{x - 2}$ at $x = 2$.
2. $f(x) = \dfrac{x}{|x|}$ at $x = 0$.
3. $f(x) = \dfrac{x^2}{x^2 - 4}$ at $x = 2$
4. $f(x) = \dfrac{x^2 - 2x - 3}{x^2 + 6x + 9}$ at $x = -3$
5. Let $f(x) = |x| - x$.
   (a) Does $\lim_{x \to 0} f(x)$ exist? (b) Is $f(x)$ continuous at x=0?
6. Discuss the continuity of each of the following functions.
   (a) $f(x) = \dfrac{1}{x}$.
   (b) $g(x) = \dfrac{x^2 - 1}{x + 1}$.
   (c) $h(x) = \sqrt{x}$.
7. Show that, if possible, there exists c in the interval such that $f(c) = k$.
   (a)$f(x) = x^2 - 2x + 1; k = 4, [-3, 2]$.
   (b)$f(x) = \dfrac{1}{2x - 1}; k = 0.5, [1, 2]$.
   (c)$f(x) = 20 \sin(x + 3) \cos(\dfrac{x^2}{2}); k = -10, [0, 5]$.
8. (a) Given that $3 - x^2 \leq u(x) \leq 3 + x^2$ for all $x \neq 0$, find $\lim_{x \to 0} u(x)$.
   (b) If $\lim_{x \to a} |f(x)| = 0$, find $\lim_{x \to a} f(x)$.

Find the limit of the following questions.

9. $\lim_{x\to 2} \frac{x-\sqrt{2+x}}{x-2}$.

10. $\lim_{x\to 1^-} \frac{x^2+2x-3}{|x-1|}$.

11. $\lim_{x\to 2} \frac{x^2+4x-12}{|x-2|}$.

12. $\lim_{\theta\to 0} \frac{\sin\theta}{\theta}$.

13. $\lim_{\theta\to 0} \frac{1-\cos^2(\theta)}{\theta^2}$.

14. $\lim_{x\to +\infty} \sqrt{x^2+x+1}-x$.

15. $\lim_{x\to\infty} \frac{4x^2-x+3}{3x^2+5}$.

16. $\lim_{x\to\infty} \frac{4x-3}{3x^2+5}$.

17. $\lim_{x\to\infty} \frac{x-\cos(x)}{x}$

18. $\lim_{x\to\infty} \frac{4x^2-3}{3x+5}$.

19. $\lim_{x\to\infty} (\sqrt{x^2+x}-x)$.

20. $\lim_{x\to 2} \frac{x+1}{x-2}$.

21. $\lim_{x\to 1} \frac{x^2-1}{x^2-3x+2}$.

22. $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$.

23. $\lim_{x\to\infty} \frac{x^2+3}{2^x}$.

24. $\lim_{x\to +\infty} x(e^{\frac{2}{x}}-1)$.

25. $\lim_{x\to +\infty} xe^{-3x}$.

26. $\lim_{x\to +\infty} \frac{x^2+3x+1}{x\ln x}$.

27. $\lim_{x\to 1^+} \sqrt[3]{x+1}\ln(x+1)$.

28. $\lim_{x\to\infty} -(x+1)(e^{\frac{1}{x+1}}-1)$.

29. $\lim_{x\to +\infty} \frac{tan^{-1}(3x)-\frac{\pi}{2}}{tan^{-1}(7x)-\frac{\pi}{2}}$.

Chapter 10

Differentiation

10.1 Introduction

This section deals with the problem of finding a straight line L that is tangent to a curve C at a point P. Let C be the graph of $y = f(x)$, and P be the point (x_0, y_0) on C so that $y_0 = f(x_0)$. Also assume that P is not an endpoint of C.

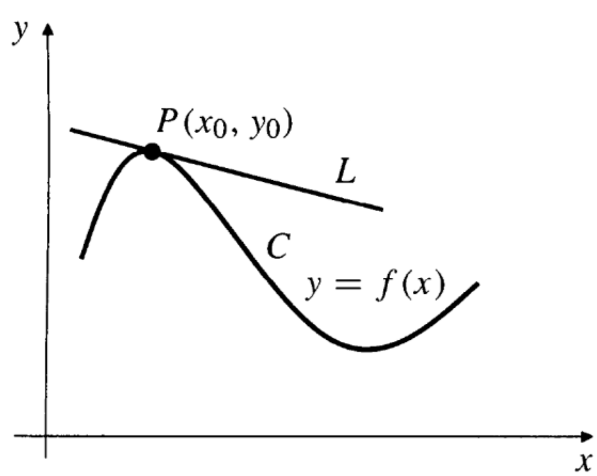
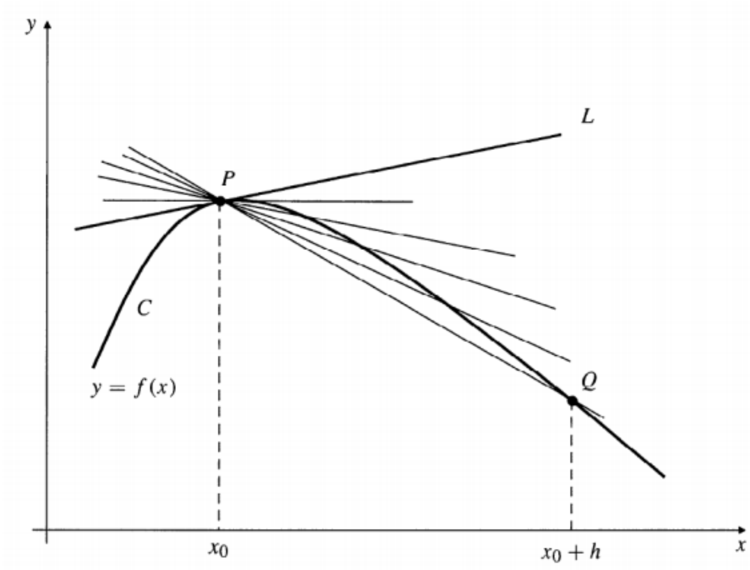


Figure 2.1 L is tangent to C at P



Given the curve $y = f(x)$ and the point $P = (x_0, f(x_0))$, we choose a difference point Q on the curve C so that $Q = (x_0 + h, f(x_0 + h))$. Note that h can be positive or negative, depending on whether Q is to the right or left of P.

The line through P and Q is called a **secant line** to the curve. The slope of the line PQ , also known as **difference quotient**, is

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

.

For **nonvertical tangent line**,
Suppose that the function f is continuous at $x = x_0$ and that

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = m$$

exists. Then the straight line having slope m and passing through the point $P = (x_0, f(x_0))$ is called the **tangent line** to the graph of $y = f(x)$ at P. An equation of this tangent line is

$$y = m(x - x_0) + y_0$$

For **vertical tangents**,
If f is continuous at $P = (x_0, y_0)$, where $y_0 = f(x_0)$, and if either

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \infty$$

or

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=-\infty$$

then the vertical line $x=x_0$ is tangent to the graph $y=f(x)$ at P.
For **no tangent line**,
If the limit of the difference quotient fails to exist in any other way than by being ∞ or $-\infty$, the graph $y=f(x)$ has no tangent line at P.

Example 10.1. Find an equation of the tangent line to the curve $y=x^2$ at the point (1,1).
Note that $f(x)=x^2, x_0=1$ and $y_0=f(x_0)=1$. By definition, the slope of the tangent line is

$$\begin{aligned} m &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} \\ &= \lim_{h\to 0}\frac{(1+h)^2-1}{h} \\ &= \lim_{h\to 0}(2+h) \\ &= 2 \end{aligned}$$

Therefore, the equation of the tangent line is

$$y=m(x-x_0)+y_0=2(x-1)+1=2x-1$$

▲

Example 10.2. Find the tangent line to the curve $y=x^{\,dfrac13}$ at the point $(0,0)$.
For this curve, the limit of the quotient for f at $x=0$ is

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{h^{1/3}}{h}=\lim_{h\to 0}\frac{1}{h^{2/3}}=\infty$$

Thus by definition, the vertical line $x=0$ (i.e. the y-axis) is the tangent line to the curve $y=x^{1/3}$ at the point $(0,0)$.

▲

Example 10.3. Does the graph of $y=|x|$ have a tangent line at $x=0$?
For this curve, the difference quotient is

$$\frac{f(x_0+h)-f(x_0)}{h}=\frac{|0+h|-|0|}{h}=\frac{|h|}{h}$$

Now since $\dfrac{|h|}{h}$ has different right and left limits at 0, the Newton quotient has no limit at $h\to 0$. This implies that $y=|x|$ has no tangent line at $(0,0)$. ▲

10.2 Derivative

From the Newton quotient rule, we can deduce the following definition.

10.2.1 Definition

The **slope** of a curve C at a point P is the slope of the tangent line to C at P if such a tangent line exists. In particular, the slope of the graph of $y = f(x)$ at the point x_0 is

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

We also call it the **derivative** of a function $f = C$ at x_0, also known as f'.
For the derivative of f at all points x,

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

If $f'(x)$ exist (i.e., is a finite real number), we say that f is **differentiable** at x.

Example 10.4. Find the slope of the curve $y = \dfrac{x}{2x+3}$ at the point $x = 3$.

The slope of the curve at $x = 3$ is

$$\begin{aligned} m &= \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} \\ &= \lim_{h \to 0} \frac{\dfrac{3+h}{9+2h} - \dfrac{1}{3}}{h} \\ &= \lim_{h \to 0} \frac{1}{27 + 6h} \\ &= \frac{1}{27} \end{aligned}$$

▲

Example 10.5. Show that if $f(x) = ax + b$, then $f'(x) = a$.

By definition, we have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{a(x + h) + b - (ax + b)}{h} \\ &= \lim_{h \to 0} \frac{ah}{h} \\ &= a. \end{aligned}$$

▲

10.3 Differentiation Rule

There are some fundamental derivative rules that we should pay attention to.

10.3.1 Linear Rules of Differentiation

If f and g are differentiable at x, and if c is a constant, then the functions $f+g$, $f-g$ and cf are all differentiable at x and

$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$

$$(cf(x))' = cf'(x)$$

10.3.2 Product Rule of Differentiation

If f and g are differentiable at x, then their product fg is also differentiable at x, and

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

Example 10.6. Find the derivative of $(x^2+1)(x^3+4)$ using the Product rule.

Using the Product rule, we have

$$\begin{aligned}\frac{d}{dx}(x^2+1)(x^3+4) &= 2x(x^3+4)+(x^2+1)(3x^2)\\ &= 5x^4+3x^2+8x\end{aligned}$$

▲

10.3.3 Quotient Rule of Differentiation

If f and g are differentiable at x, and if $g(x) \neq 0$, then the quotient $\frac{f}{g}$ is differentiable at x and

$$(\frac{f(x)}{g(x)})' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

10.3.4 Reciprocal Rule

If g is differentiable at x and if $g(x) \neq 0$, then the reciprocal $1/g$ is also differentiable at x and

$$(\frac{1}{g(x)})' = \frac{-g'(x)}{g^2(x)}$$

Example 10.7. Find the derivatives of $y = \frac{x^2}{x+1}$.

By the Quotient rule, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2)'(x+1) - x^2(x+1)'}{(x+1)^2}\\ &= \frac{2x(x+1)-x^2}{(x+1)^2}\\ &= \frac{x^2+2x}{(x+1)^2}\end{aligned}$$

▲

10.3.5 Elementary Derivatives

From the derivative rule, we can derive the derivatives of some elementary functions.

f(x)	f'(x)
c	0
x	1
x^2	$2x$
$\frac{1}{x}$	$-\frac{1}{x^2} \quad (x \neq 0)$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}} \quad (x > 0)$
x^r	$rx^{r-1} \quad (r \neq 0 \text{ and } x^{r-1} \text{ is real})$
$\lvert x \rvert$	$sgn(x) = \frac{x}{\lvert x \rvert}$

10.4 Derivative of Trigonometry

The derivatives of trigonometry are shown as follows.

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\cos(x) = \sin(x)$$

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

Proof 10.1. To prove that $\frac{d}{dx}\sin x = \cos x$,

$$\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{2\cos(\frac{x+h+x}{2})\sin(\frac{x+h-x}{2})}{h} \\
&= \lim_{h\to 0}\frac{2\cos(\frac{x+h+x}{2})\frac{1}{2}\lim_{h\to 0}\sin(\frac{h}{2})}{\frac{h}{2}} \\
&= \cos(\frac{2x}{2})\cdot 1 \\
&= \cos x
\end{aligned}$$

■

10.5 Derivative of Natural Log and Exponential

For $f(x) = \ln(x)$, the derivative is

$$f'(x) = \frac{1}{x}$$

For $g(x) = e^x$, the derivative is the same, i.e.

$$g'(x) = e^x$$

Therefore, if you differentiate the function infinitely many times, the outcome should still be the same.

10.6 Chain Rule

If $f(u)$ is differentiable at $u = g(x)$, and $g(x)$ is differentiable at x, then the composite function $f \circ g(x) = f(g(x))$ is differentiable at x and

$$(f(g(x)))' = f'(g(x))g'(x).$$

In *Leibniz*'s notation, by letting $y = f(u)$ where $u = g(x)$, we have $y = f(g(x))$ and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

The Chain rule can be extended to composite functions with any finite number of intermediate functions. Assume that $y = f(u), u = g(v)$ and $v = h(x)$. Then $y = f(g(h(x)))$ and the Chain rule is

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}.$$

Example 10.8. Find the derivative of $y = \dfrac{1}{x^2 - 4}$

By the Chain rule, we let $y = f(g(x))$ with $f(u) = \frac{1}{u}$ and $u = g(x) = x^2 - 4$. Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \frac{-1}{u^2}(2x) \\ &= \frac{-2x}{(x^2 - 4)^2} \end{aligned}$$

▲

10.7 High-order Derivatives

The second derivative of a function is the derivative of its derivative. If $y = f(x)$, the second derivative is defined by

$$y'' = f''(x).$$

For any positive integer n, the nth derivative of a function is obtained from the function by differentiating successively n times. If the original function is $y = f(x)$, the nth derivative is denoted by

$$\frac{d^n y}{dx^n}$$

or

$$f^{(n)}(x).$$

Specially, we have

$$f^{(k)}(x) = \frac{d}{dx}[f^{(k-1)}(x)],$$

for $k = 1, 2, \cdots, n$, if $f^{(0)}(x), f^{(1)}(x), \cdots, f^{(n)}(x)$ are all differentiable.

Example 10.9. Calculate the first 3 derivatives of $f(x) = \sqrt{x^2+1}$.
Note that $f(x) = (x^2+1)^{\frac{1}{2}}$, we have

$$\begin{aligned}
f'(x) &= \frac{1}{2}(x^2+1)^{\frac{-1}{2}} \\
&= x(x^2+1)^{\frac{-1}{2}} \\
f"(x) &= (x^2+1)^{\frac{-1}{2}} + x\frac{-1}{2}(x^2+1)^{\frac{-3}{2}}(2x) \\
&= (x^2+1)^{\frac{-3}{2}} \\
f^{(3)}(x) &= (\frac{-3}{2})(x^2+1)^{\frac{-5}{2}}(2x) \\
&= -3x(x^2+1)^{\frac{-5}{2}}
\end{aligned}$$

▲

10.8 Mean-Value Theorem

Suppose that the function f is continuous on the closed, finite interval $[a, b]$ and that it is differentiable on the open interval (a, b). Then there exists a point c in the open interval (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

This says that the slope of the chord line joining the points $(a, f(a))$ and $(b, f(b))$ is equal to the slope of the tangent line to the curve $y = f(x)$ at the point $(c, f(c))$, so the two lines are parallel.

Example 10.10. Show that $\sin(x) < x$ for all $x > 0$.

Case (i): If $x > \frac{\pi}{2}$, then $\sin(x) \leq 1 \leq \frac{\pi}{2} < x$.

Case (ii): If $0 < x \leq \frac{\pi}{2}$, then by the Mean-Value theorem, there exists a point $c \in (0, \frac{\pi}{2})$ such that

$$\begin{aligned} \frac{\sin(x)}{x} &= \frac{\sin(x)-\sin(0)}{x-0} \\ &= \frac{d}{dx}\sin(x)|_{x=c} \\ &= \cos(c) < 1. \end{aligned}$$

By (i) and (ii), $\sin(x) < x$ for all $x > 0$. ▲

10.9 Exercises

Find the derivatives of question 1 to 18 and the second derivatives of question 19 to 37 with respect to x.

1. $y = (1-3x)^4$.
2. $y = (x^2+3x-5)^5$.
3. $y = (x^3-2+\frac{1}{x^2})^6$.
4. $y = \sqrt{x^7}+\sqrt{x^5}+\sqrt{x}$.
5. $y = \frac{1}{\sqrt{x^2+5x-3}}$.
6. $y = (2x-3)(4x-7)^3$.
7. $y = \frac{(3x-4)^2}{(2x-1)^3}$.
8. $y = \sqrt{x-\sqrt{x^2-3}}$.
9. $y = \frac{2\sqrt{4-3x}}{(x^2+6)^3}$.
10. $y = \sqrt{(1-2x)^3}\sqrt[3]{3x^2+1}$.
11. $y = (3x-\sqrt{9x^2-8})^5$.
12. $y = \ln(x^2+1)$.
13. $y = e^{x/2}$.
14. $y = \ln(\frac{1}{x-1})$.
15. $y = \ln(\sqrt{x}-3)$.
16. $y = y = x^2\ln(2x^3+1)$.
17. $y = \ln\frac{x^2+1}{\sqrt{x}}$.
18. $y = \ln\frac{5x}{x+1}$.
19. $y = \ln(\sin 3x)$.
20. $y = \frac{4x}{(\ln x)^3}$.
21. $y = \ln(\tan e^{2x})$.
22. $y = \frac{e^{sin^2 x}}{e^{\cos^2 x}}$.
23. $y = 2x^2-6x+7$.
24. $y = 3x-\frac{1}{12x^3}+7$.
25. $y = x(4x-3)^2$.
26. $y = \cos 5x$.
27. $y = \tan^2 x$.
28. $y = \frac{1}{x^2-1}$.
29. $y = \frac{\sin x}{\sin x+1}$.
30. $y = \sqrt{x+1}-\frac{1}{\sqrt{x+1}}$.
31. $y = 2x\ln x^3$.
32. $y = \sin^3 2x$.
33. $y = x^2e^x$.
34. $y = \sin x\cos 3x$.
35. $y = \ln(3x^2-2x+1)$.
36. $y = x^{2x}\sin x$.
37. $y = \sqrt{x^5-4x^2+1}\cdot 3x^2$.

38. Let $y = 4x^2 + 2x + 1$. Prove that $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + 2 = 0$.

39. Let $y = \sqrt{x^2 - 2}$. Prove that $(\dfrac{dy}{dx})^2 + y\dfrac{d^2y}{dx^2} = 1$.

40. Let $y = \dfrac{\sin x}{x}$. Find the value of k such that $x\dfrac{d^2y}{dx^2} + dy + k\dfrac{dy}{dx} = 0$.

41. Determine all the number(s) c which satisfy the conclusion of Mean-Value Theorem for the given function and interval.

    (a) $f(x) = x^2 - 2x - 8$ on $[-1, 3]$.

    (b) $f(x) = 2x - x^2 - x^3$ on $[-2, 1]$.

    (c) $f(x) = 4x^3 - 8x^2 - 2$ on $[2, 5]$.

    (d) $f(x) = 8x + e^{-3x}$ on $[-2, 3]$.

42. Suppose we know that $f(x)$ is continuous and differentiable on the interval $[-7, 0]$, that $f(-7) = -3$ and that $f'(x) \leq 2$. What is the largest possible value for $f(0)$?

43. Show that $f(x) = x^3 - 7x^2 + 25x + 8$ has exactly one real root.

Chapter 11

Application of Differentiation

11.1 L'Hôpital Rule

There are some mathematical "black holes" that we do not understand, but we wish to. And L'Hôpital Rule is a super powerful tool to tackle such problems. It can be so useful when the following **indeterminate forms** happen: [‡]

Type	Example
$[0/0]$	$\lim_{x\to 0} \frac{\sin x}{x}$
$[\infty/\infty]$	$\lim_{x\to 0} \frac{\ln(1/x^2)}{\cot(x^2)}$
$[0 \cdot \infty]$	$\lim_{x\to 0+} x \ln \frac{1}{x}$
$[\infty - \infty]$	$\lim_{x\to (\pi/2)-} \left(\tan x - \frac{1}{\pi - 2x}\right)$
$[0^0]$	$\lim_{x\to 0+} x^x$
$[\infty^0]$	$\lim_{x\to (\pi/2)-} (\tan x)^{\cos x}$
$[1^\infty]$	$\lim_{x\to \infty} \left(1 + \frac{1}{x}\right)^x$

Now we know that L'Hôpital Rule shall be used when a indeterminate form happens. So how to use the method? Technically, L'Hôpital Rule tells us that if we have an indeterminate form, all we need to do is differentiate the numerator and the denominator seperately, then take the limit. Take a look at the following example.

Example 11.1. Find $\lim_{x\to -2} \frac{x+2}{\ln(x+3)}$.

[‡]http://tutorial.math.lamar.edu/Classes/CalcI/LHospitalsRule.aspx

We have $\lim_{x\to -2} \frac{x+2}{\ln(x+3)}$ which is in $[\frac{0}{0}]$ form. Therefore, by L'Hôpital Rule,

$$\begin{aligned}\lim_{x\to -2} \frac{x+2}{\ln(x+3)} &= \lim_{x\to -2} \frac{1}{\frac{1}{x+3}} \cdot (1+0) \\ &= \lim_{x\to -2} \frac{1}{\frac{1}{x+3}} \\ &= \lim_{x\to -2} x+3 \\ &= -2+3 = 1\end{aligned}$$

▲

Example 11.2. Calculate

$$\lim_{x\to\infty} (1+x)^{\frac{1}{x}}.$$

Since $\lim_{x\to\infty} (1+x)^{\frac{1}{x}}$ is of type $[\infty^0]$, set

$$y = (1+x)^{\frac{1}{x}}.$$

So that

$$\ln y = \ln(1+x)^{\frac{1}{x}} = \frac{\ln(1+x)}{x}$$

$$\begin{aligned}\lim_{x\to\infty} \ln y &= \lim_{x\to\infty} \frac{\ln(1+x)}{x} [\frac{\infty}{\infty}] \\ &= \lim_{x\to\infty} \frac{1}{1+x} \\ &= 0\end{aligned}$$

Taking exponentials gives

$$\lim_{x\to\infty} y = e^0 = 1.$$

$$\lim_{x\to\infty} (1+x)^{\frac{1}{x}} = 1.$$

▲

11.2 Tangents and Normals

Let C be a curve on a rectangular coordinate plane. Consider a point (x, y) on C. The derivative $\frac{dy}{dx}$ is the slope of the **tangent** to the curve at this point. For the slope of the tangent at a particular point $P(x_1, y_1)$, we simply put the coordinates in the expression of $\frac{dy}{dx}$ and that value of $\frac{dy}{dx}$ is denoted by $\frac{dy}{dx}|_{(x_1,y_1)}$, which is actually the slope m of the tangent to the curve at point P.

It is easy to determine the equations of the tangent and the normal to the curve at point P. By the point-slope form, we have
Equation of the tangent at P:

$$y - y_1 = m(x - x_1).$$

Equation of the normal at P:

$$y - y_1 = -\frac{1}{m}(x - x_1), \text{where } m \neq 0$$

.

Example 11.3. Find the equations of the tangent and the normal to the curve $y = x^2 + 2x + 1$ at the point $(0, 1)$.
First, differentiate both sides of the equation with respect to x,

$$y' = 2x + 2$$

Then, substitute the point $(0, 1)$ into the derivative,

$$y'|_{(0,1)} = 2(0) + 2 = 2$$

which is the slope of the tangent.
Therefore, the equation of the tangent to the curve at $(0, 1)$ is

$$\begin{aligned} y - 1 &= 2(x - 0) \\ 2x - y + 1 &= 0 \end{aligned}$$

Slope of the normal to the curve at $(0, 1) = -\frac{1}{2}$.

Therefore, the equation of the normal to the curve at $(0, 1)$ is

$$\begin{aligned} y - 1 &= -\frac{1}{2}(x - 0) \\ x + 2y - 2 &= 0 \end{aligned}$$

▲

11.3 Find Maximum and Minimum

11.3.1 Absolute Extreme Points

A function f has an **absolute maximum value** $f(x_0)$ at the point x_0 in its domain if $f(x) \leq f(x_0)$ holds for every x in the domain of f.

Similarly, the function f has an **absolute minimum value** $f(x_1)$ at the point x_1 in its domain if $f(x) \geq f(x_1)$ for every x in the domain of f.

Note

- A function can have at most one absolute maximum or minimum value, although this value can be assumed at many points.

- Maximum and minimum values of a function are collectively referred to as **extreme values**.

- A function may not have any absolute extreme value. E.g.: $f(x) = \dfrac{1}{x}$ has no finite absolute maximum.

11.3.2 Local Extreme Points

A function f has a **local maximum value** $f(x_0)$ at the point x_0 if $f(x) \leq f(x_0)$ in a certain neighborhood of x_0.

Similarly, the function has a **local minimum value** $f(x_1)$ at the point x_1 if $f(x) \geq f(x_1)$ in a certain neighborhood of x_1.

11.3.3 Increasing and Decreasing Functions

Let J be an open interval, and let l be an interval consisting of all points in J and possibly one or both of the endpoints of J.

Suppose that f is continuous on I and differentiable on J.

1. If $f'(x) > 0$ for all x in J, then f is **strictly increasing** on I.

2. If $f'(x) < 0$ for all x in J, then f is **strictly decreasing** on I

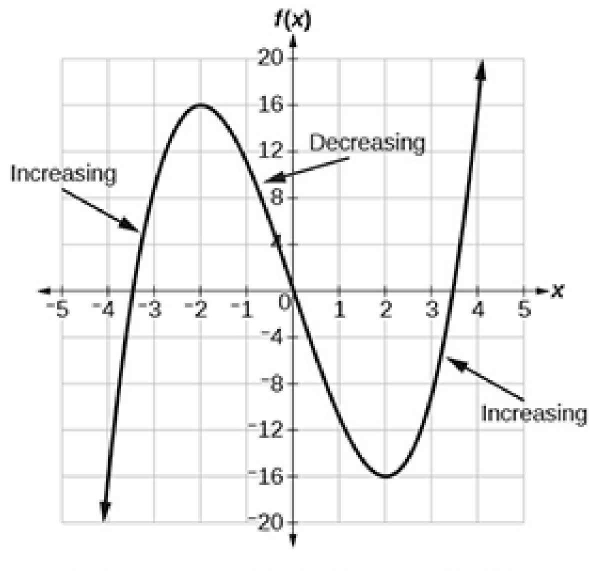
3. If $f'(x)' \geq 0$ for all x in J, then f is **nondecreasing** on I

4. If $f'(x) \leq 0$ for all x in J, then f is **nonincreasing** on I

Example 11.4. For an instance, $y = 3x + 5$ is a strictly increasing function for $x \in \mathbb{R}$ since $y' = 3$ which is greater than zero. ▲

There are two methods to find out the maximum or minimum point of a curve of f.

First Derivative Test



From the curve above, as x increases through -2, the function $f(x)$ is first increasing and then decreasing. We say $f(x)$ has a maximum point at $x = -2$ and thus $(-2, 16)$ is called a maximum point of the curve $y = f(x)$.

Similarly, as x increases through 2, the function $f(x)$ is first decreasing and then increasing. We say $f(x)$ has a minimum at $x = 2$ and thus $(2, -16)$ is called a minimum point of the curve $y = f(x)$.

Example 11.5. Consider the curve $y = 2x^3 - 3x^2 - 12x + 10$.

(a) Find the stationary points of the curve.

(b) Find the range of values of x such that y is decreasing or increasing.

(c) Hence, find the maximum and minimum points of the curve.

(a) First we find the first derivative.

$$\frac{dy}{dx} = 6(x+1)(x-2)$$

Then put $\frac{dy}{dx} = 0$,

$$\begin{aligned} 6(x+1)(x-2) &= 0 \\ x &= -1 \text{ or } 2 \end{aligned}$$

When $x = -1, y = 17$. When $x = 2, y = -10$. Therefore the stationary points are $(-1, 17)$ and $(2, -10)$.

(b) The trend of the curve is as shown below.

x	x < -1	x = -1	-1 < x < 2	x = 2	x > 2
y'	Increasing	0	Decreasing	0	Increasing

(c) As x increases through -1, $\frac{dy}{dx}$ changes sign from positive to negative. Hence $(-1, 17)$ is a maximum point.

As x increases through 2, $\frac{dy}{dx}$ changes sign from negative to positive. Hence $(2, -10)$ is a minimum point.

▲

Second Derivative Test

In the First Derivative Test, we first find the stationary points by solving $f'(x) = 0$. Then we consider the changes in sign of $f'(x)$, i.e. the slope of the tangent, as x increases through the x-coordinate of a stationary point. In Second Derivative Test, we have the following testing method.

Suppose $f(x)$ and $f''(x)$ are differentiable with respect to x in an interval containing $x = a$.
If $f'(a) = 0$ and $f''(a) < 0$, then $f(x)$ has a (local) maximum at $x = a$.
If $f'(a) = 0$ and $f''(a) > 0$, then $f(x)$ has a (local) minimum at $x = a$.

Example 11.6. Given the curve $y = (x+1)^2(x+4)$, find the maximum and minimum points of the curve.
First we find the first and second derivatives of the equation.

$$\begin{aligned} \frac{dy}{dx} &= 3(x+1)(x+3) \\ \frac{d^2y}{dx^2} &= 6(x+2) \end{aligned}$$

Put $\frac{dy}{dx} = 0$,

$$\begin{aligned} 3(x+1)(x+3) &= 0 \\ x &= -1 \text{ or } -3. \end{aligned}$$

When $x = -1, y = 0$. When $x = -3, y = 4$.
Therefore the stationary points are $(-1, 0)$ and $(-3, 4)$.

For $(-1, 0)$, $\frac{d^2y}{dx^2}|_{x=-1} = 6 > 0$. Therefore, $(-1, 0)$ is a minimum point.

For $(-3, 4)$, $\frac{d^2y}{dx^2}|_{x=-3} = -6 < 0$. Therefore, $(-3, 4)$ is a maximum point.

▲

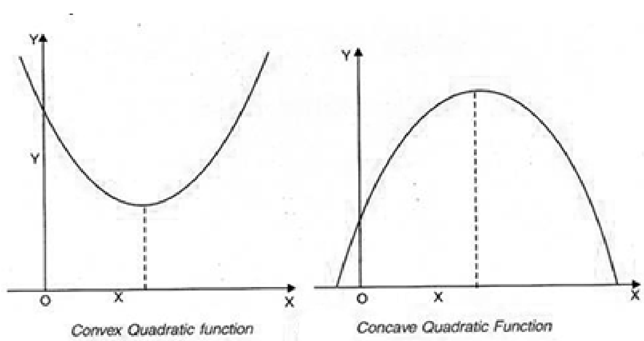
11.4 Curve Sketching

Before we start curve sketching, there are a few fundamental concepts that we should know.

11.4.1 Concavity and Inflections

We say that the function f is **concave up** on an open interval I if it is differentiable there and the derivative f' is an increasing function on I.

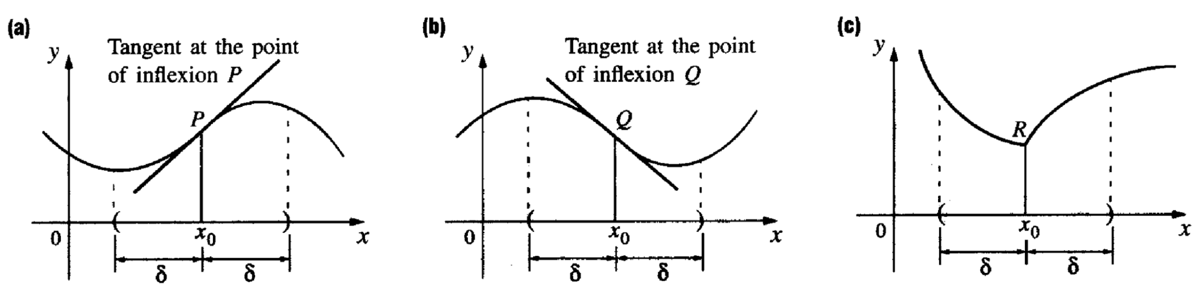
Similarly, f is **concave down** on I if f' exists and is an decreasing function on I.



If $f(x)$ is a function on $[a, b]$ such that $f(x)$ is second differentiable on (a, b), then
(i) $f"(x) > 0$ if and only if $f(x)$ is concave upward on (a, b)
(ii) $f"(x) < 0$ if and only if $f(x)$ is concave download on (a, b).

11.4.2 Point of Inflection

Let $f(x)$ be a continuous function. A point $(a, f(a))$ on the graph of $f(x)$ is a point of inflection if the graph on one side of this point is concave downward and concave upward on the other side. That is, the graph changes concavity at $x = a$.



Note A point of inflection of a curve $y = f(x)$ must be a continuous point but need not be differentiable there. In Figure (c), R is a point of inflection of the curve but the function is not differentiable at x_0.

Example 11.7. Find the points of inflection of the curve $y = x^4 - 6x^2 + 8x + 10$.

$$\begin{aligned} y' &= 4x^3 - 12x + 8 \\ y" &= 12x^2 - 12 \\ &= 12(x-1)(x+1) \end{aligned}$$

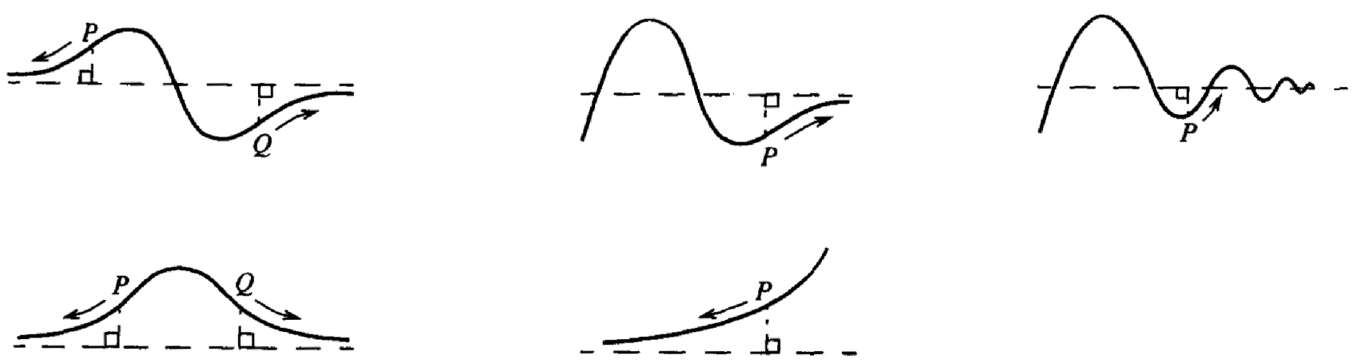
For $y" = 0, x = 1$ or -1.

x	x < −1	x = −1	−1 < x < 1	x = 1	x > 1
y''	+	0	-	0	+
y	concave upward	pt of inflexion	concave downward	pt of inflexion	concave upward

The curve has points of inflection at $x = 1$ and $x = -1$. These two points are $(-1, -3)$ and $(1, 13)$. ▲

11.4.3 Asymptote

A straight line is an **asymptote** to a curve if and only if the perpendicular distance from a variable point on the curve to the line approaches to zero as a limit when the point tends to infinity along the curve on both sides or one side of the curve. (see figure below)



(i) the line $x = c$ is said to be vertical asymptote of the curve $y = f(x)$ if

$$\lim_{x \to c^+} f(x) = \infty \text{ or } \lim_{x \to c^-} f(x) = \infty$$

(ii) the line $y = mx + c$ is said to be an oblique/horizontal asymptote of the curve $y = f(x)$ if

$$\lim_{x \to \infty} [f(x) - (mx + c)] = 0$$

To find the asymptote $y = mx + c$,

$$m = \lim_{x \to \infty} \frac{f(x)}{x} \text{ and } c = \lim_{x \to \infty} [f(x) - mx]$$

Example 11.8. (i) The curve $y = \frac{1}{x}$ has two asymptotes $x = 0$ or $y = 0$. (ii) The curve $y = e^x$ has an asymptote $y = 0$. (iii) The curve $y = \frac{1}{x}\sin x$ has an asymptote $y = 0$. ▲

11.4.4 Procedures

Finally, we can sketch a curve by using the knowledge mentioned above. Steps of skecthing a graph involve

1. Find the first derivative and second derivative of $f(x)$.
2. Determine the range of x that it is increasing or decreasing.
3. Determine the relative extreme points and points of inflection of $f(x)$.
4. Find the asymptotes of the graph of $f(x)$.
5. Finally, sketch the graph.

Example 11.9. Sketch the curve $y = \frac{x^2}{2-x}$.
Let $f(x) = \frac{x^2}{2-x}$. First, find the first and second derivative of the function.

$$\begin{aligned} f'(x) &= \frac{4x - x^2}{(2-x)^2} \\ f''(x) &= \frac{8}{(2-x)^3} \end{aligned}$$

To find the extreme point(s) of the curve, put $f'(x) = 0$.

$$\begin{aligned} \frac{4x - x^2}{(2-x)^2} &= 0 \\ x(4-x) &= 0 \\ x &= 0 \text{ or } 4 \end{aligned}$$

When $x = 0, y = 0$. When $x = 4, y = -8$.
Therefore, the extreme points are $(0, 0)$ and $(4, -8)$.
For $(0, 0)$, $f''(0) = \frac{8}{(2-0)^3} = 1 > 0$. Therefore, $(0, 0)$ is a minimum point.
For $(4, -8)$, $f''(4) = \frac{8}{(2-4)^3} = -1 < 0$. Therefore, $(4, -8)$ is a maximum point.

To find the concavity of the curve,

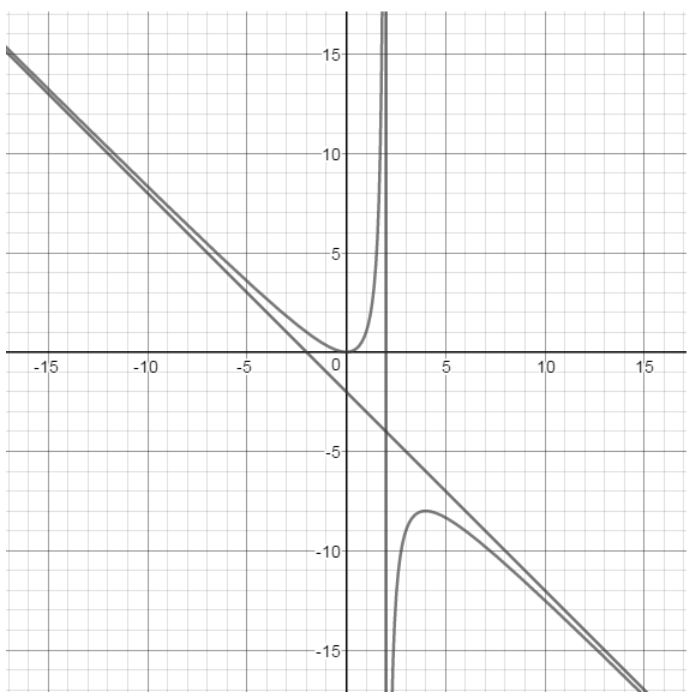
x	**x < 0**	**x = 0**	**0 < x < 2**	**x = 2**	**2 < x < 4**	**x = 4**	**x > 4**
y'	**Decreasing**	**0**	**Increasing**	**Undefined**	**Increasing**	**0**	**Decreasing**
y''	+	/	+	/	-	/	-
y	**Concave upward & decreasing**	**Local Minimum**	**Concave upward & increasing**	/	**Concave downward & increasing**	**Local Maximum**	**Concave download & decreasing**

To find the asymptote(s),note that $f(x)$ is defined for all real values of x except $x = 2$. Therefore, the curve is discontinuous at $x = 2$ and thus $x = 2$ is a vertical asymptote of the graph.
For the oblique/horizontal asymptote, let $y = mx + c$ be the asymptote. Then the slope and the constant term are

$$\begin{aligned} m &= \lim_{x\to\infty} \frac{\frac{x^2}{2-x}}{x} \\ &= \lim_{x\to\infty} \frac{x}{2-x} \\ &= \lim_{x\to\infty} \frac{1}{\frac{2}{x}-1} \\ &= \frac{1}{0-1} \\ &= -1. \end{aligned}$$

$$\begin{aligned} c &= [\lim_{x\to\infty} f(x) - mx] \\ &= [\lim_{x\to\infty} \frac{x^2}{2-x} + x] \\ &= [\lim_{x\to\infty} \frac{x^2}{2-x} + \frac{2x-x^2}{2-x}] \\ &= \lim_{x\to\infty} \frac{2x}{2-x} \\ &= \frac{2}{0-1} \\ &= -2. \end{aligned}$$

Therefore, the oblique asymptote is $y = -x - 2$.
To find the x-intercept and y-intercept, sub $x = 0$ and $y = 0$ into the equation. When $x = 0$, $y = 0$. Therefore, y-intercept is 0. When $y = 0$, $\frac{x^2}{2-x} = 0$, i.e. $x = 0$. Therefore, x-intercept is 0.
With the above information, the curve is sketch below.



▲

11.5 Exercises

For Question $1-10$,, find the equations of the tangent and the normal to the curve at the given point.

1. $y = x^3 + 4x - 7$ at the point $(1, -2)$
2. $y = (2x^2 - 3)^3$ at the point $(-1, -1)$
3. $y = -\dfrac{2}{3x-5}$ at the point $(1, 1)$
4. $y = \sqrt{2x}$ at the point $(2, 2)$
5. $y = x\sqrt{x+1}$ at the point $(0, 0)$
6. $y = \cos 4x$ at the point $\dfrac{\pi}{8}, 0)$
7. $x^2 + y^2 = 13$ at the point $(2, 3)$
8. $y^2 = -5x^2 + x + 9$ at the point $(0, 3)$
9. $x^2y - xy^2 + 16 = 0$ at the point $(2, 4)$
10. $x^3 - 3xy + y^3 = 13$ at the point $(-1, 2)$

11. The curve $C : 3x^2 + 2xy - y^2 = 7$ is given.
    (a) Find $\dfrac{dy}{dx}$.
    (b) Hence, find the equations of the two tangents to the curve C which are parallel to the line $3y + 5x = 0$.

12. Consider the curve $C : x^2 - 2y\sin x - y^2 = \pi$.
    (a) Find $\dfrac{dy}{dx}$.
    (b) $P(\dfrac{\pi}{2}, -\dfrac{\pi}{2})$ is a point on the curve C. Find the equation of the tangent to the curve at P.

13. Consider the curve $C : x^2y - xy^2 + 4 = 0$.
    (a) Find $\dfrac{dy}{dx}$.
    (b) If the line $x + 2y = \sqrt{3} - 5$ is the normal to the curve C at a point P, find the coordinates of P.

14. Find the equation of the normal to the curve $y(x+1)^2 = 2$ which is perpendicular to the line $y = -\dfrac{1}{2}x$.

15. Find the maximum and minimum points of each of the following curves by the First Derivative Test.
    (a) $y = 2x^2 - 4x + 3$
    (b) $y = -x^2 + 6x + 5$
    (c) $y = 3x - x^3$
    (d) $y2x^3 - 6x^2 + 5$
    (e) $y = 4x^3 - 3x^2 - 18x + 6$
    (f) $y = (x-2)(x^2 - 4x + 1)$
    (g) $y = x + 2\sin x$ for $0 \le x \le 2\pi$
    (h) $y = \sin x + \dfrac{1}{2}\sin 2x$ for $0 \le x \le \pi$

16. Find the maximum and minimum points of each of the following curves by the Second Derivative Test.
    (a) $y = x^4 - 2x^2 - 8$

(b) $y = 3x^4 - 8x^3 - 6x^2 + 24x + 7$

(c) $y = (x+1)^2(x-1)$

(d) $y = \frac{x^2}{x-1}$

(e) $y = 2x^2 + \frac{4}{x}$

(f) $y = \frac{1}{2}\cos 2x$ for $0 \le x \le \pi$.

(g) $y = x - e^x$

(h) $y = \ln 4x - 2x^2$

17. Consider the curve $y = \frac{x}{x^2+1}$.

(a) Find the maximum and minimum points of the curve.

(b) Find the points of inflection of the curve.

(c) Find the asymptote to the curve.

(d) Sketch the curve.

18. Consider the curve $y = x - 1 + \frac{4}{x+2}$.

(a) Find $\frac{dy}{dx}$.

(b) Find the turning points of the curve. Test whether each point is a maximum or a minimum point.

(c) Find the asymptotes to the curve.

(d) Sketch the curve.

19. Let $f(x) = \frac{a+bx}{4-x}$ for $x \ne 4$, where a and b are constants. The x-intercept and the y-intercept of the curve $y = f(x)$ are -2 and 4 respectively.

(a) Find the values of a and b.

(b) Find all the asymptotes to the curve $y = f(x)$.

(c) Sketch the curve $y = f(x)$.

20. Consider the curve $y = \frac{x^2+x+2}{1-x}$.

(a) Find the maximum and minimum points of the curve.

(b) Find the asymptotes to the curve.

(c) Sketch the curve.

21. **HKALE 2003II 8**
‘ Let $f(x) = x^2 - \frac{8}{x-1} (x \ne 1)$.

(a) Find $f'(x)$ and $f''(x)$.

(b) Determine the values of x such that

i. $f'(x) > 0$

ii. $f'(x) < 0$

iii. $f''(x) > 0$

iv. $f''(x) < 0$

(c) Find the relative extreme point(s) and point(s) of inflection of $f(x)$.

(d) Find the asymptotes of the graph of $f(x)$.

(e) Sketch the graph of $f(x)$.